# ON SOME MINIMAX LOCATION PROBLEMS USING RECTILINEAR DISTANCE

By

PERINO MARCELLIN DEARING, JR.

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA IN PARTIAL
FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1972

UNIVERSITY OF FLORIDA
3 1262 08552 5755

TO KAY

# ACKNOWLEDGMENTS

The author wishes to express his gratitude and appreciation to the members of his supervisory committee, Dr. R. L. Francis, Dr. D. W. Hearn, Dr. M. H. Moore, and Dr. H. D. Ratliff, for their assistance and guidance.

The author is especially grateful to his advisor, Dr. R. L. Francis, for suggesting the research topic and for his encouragement and patience throughout the research.

Above all, the author acknowledges the support, understanding and sacrifices of his wife, Kay, that made this graduate work possible.

This research was partially supported by the Army Research Office under Contract DAH CO 4 68C 0002.

## TABLE OF CONTENTS

Page

ACKNOWLEDGMENTS . . . . . . . . . . . . . . . . . . . . . . . . . iii

LIST OF TABLES . . . . . . . . . . . . . . . . . . . . . . . . . vi

LIST OF FIGURES . . . . . . . . . . . . . . . . . . . . . . . . . vii

ABSTRACT . . . . . . . . . . . . . . . . . . . . . . . . . . . viii

CHAPTER

1 INTRODUCTION AND OVERVIEW . . . . . . . . . . . . . . . . . 1

1.1 Introduction . . . . . . . . . . . . . . . . . . . . 1
1.2 Applications . . . . . . . . . . . . . . . . . . . . 1
1.3 Related Work . . . . . . . . . . . . . . . . . . . . 2
1.4 Overview . . . . . . . . . . . . . . . . . . . . 4

2 A ONE FACILITY PROBLEM . . . . . . . . . . . . . . . . . 7

2.1 Introduction . . . . . . . . . . . . . . . . . . . . 7
2.2 Formulation . . . . . . . . . . . . . . . . . . . . 7
2.3 Applications . . . . . . . . . . . . . . . . . . . 9
2.4 Related Work . . . . . . . . . . . . . . . . . . . 11
2.5 A Transformation and Decomposition . . . . . . . . . 12
2.6 Subproblem Development . . . . . . . . . . . . . . 16
2.7 Solution Procedure . . . . . . . . . . . . . . . . 22
2.8 Additional Properties . . . . . . . . . . . . . . . 25

3 A MULTIFACILITY PROBLEM . . . . . . . . . . . . . . . . . 33

3.1 Introduction . . . . . . . . . . . . . . . . . . . . 33
3.2 Formulation . . . . . . . . . . . . . . . . . . . . 34
3.3 Applications . . . . . . . . . . . . . . . . . . . . 38
3.4 Related Work . . . . . . . . . . . . . . . . . . . . 39
3.5 The Transformation and Decomposition . . . . . . . . 39
3.6 A Network Formulation . . . . . . . . . . . . . . . 41
3.7 Properties of the Network Problem . . . . . . . . . 46
3.8 A Solution Procedure . . . . . . . . . . . . . . . . 57
3.9 A Special Case . . . . . . . . . . . . . . . . . . 63
3.10 An Extended Problem . . . . . . . . . . . . . . . . 65

TABLE OF CONTENTS (CONTINUED)

CHAPTER Page

4 A p-CENTER LOCATION PROBLEM . . . . . . . . . . . . . . . . . . 67

4.1 Introduction . . . . . . . . . . . . . . . . . . . . . . 67
4.2 Formulation . . . . . . . . . . . . . . . . . . . . . . 68
4.3 Applications . . . . . . . . . . . . . . . . . . . . . . 75
4.4 Related Work . . . . . . . . . . . . . . . . . . . . . . 76
4.5 Problem Development . . . . . . . . . . . . . . . . . . 78
4.6 Determining the Set Covering Problems . . . . . . . . 80
4.7 A Solution Procedure . . . . . . . . . . . . . . . . . 92
4.8 Additional Observations . . . . . . . . . . . . . . . . 101

5 A ONE FACILITY PROBLEM WITH RESPECT TO A COMPACT SET . . 102

5.1 Introduction . . . . . . . . . . . . . . . . . . . . . . 102
5.2 Formulation . . . . . . . . . . . . . . . . . . . . . . 103
5.3 Applications . . . . . . . . . . . . . . . . . . . . . . 103
5.4 Related Work . . . . . . . . . . . . . . . . . . . . . . 104
5.5 Properties of a Minimax Problem . . . . . . . . . . . 104
5.6 The Transformation and Decomposition . . . . . . . . 113
5.7 Properties of the Subproblems . . . . . . . . . . . . 116
5.8 Solution Procedure . . . . . . . . . . . . . . . . . . 122

6 SUMMARY AND FUTURE RESEARCH . . . . . . . . . . . . . . . . 126

6.1 Summary . . . . . . . . . . . . . . . . . . . . . . . . 126
6.2 Future Research . . . . . . . . . . . . . . . . . . . . 127

REFERENCES . . . . . . . . . . . . . . . . . . . . . . . . . . 129

BIOGRAPHICAL SKETCH . . . . . . . . . . . . . . . . . . . . . 132

# LIST OF TABLES

Table Page

3.1 Dual Variables . . . . . . . . . . . . . . . . . . . . . . . 44

4.1 Average Execution Times for the p-Center Problem, m = 20, 15 . . . . . . . . . . . . . . . . . . . . . . . 96

4.2 Average Execution Times for the p-Center Problem, m = 10, 5 . . . . . . . . . . . . . . . . . . . . . . . 97

4.3 Data for Example 4.2.1 . . . . . . . . . . . . . . . . . . 98

# LIST OF FIGURES

Figure Page

3.1 A Network N . . . . . . . . . . . . . . . . . . . . . . 47

3.2 A Network N' . . . . . . . . . . . . . . . . . . . . . . 55

4.1 Diamonds $D(P_i,1)$ . . . . . . . . . . . . . . . . . . . . 73

4.2 Squares $S(P_i',1/\sqrt{2})$ . . . . . . . . . . . . . . . . . . 82

4.3 Optimal 3-Center for Example 4.2.1 . . . . . . . . . . . 99

4.4 Optimal 2-Center for Example 4.2.1 . . . . . . . . . . . 100

Abstract of Dissertation Presented to the Graduate Council
in Partial Fulfillment of the Requirements for
the Degree of Doctor of Philosophy

# ON SOME MINIMAX LOCATION PROBLEMS USING RECTILINEAR DISTANCE

By

Perino Marcellin Dearing, Jr.

December, 1972

Chairman: Dr. R. L. Francis
Co-Chairman: Dr. D. W. Hearn
Major Department: Industrial and Systems Engineering

A study is presented of some minimax location problems in the Euclidean plane under the assumption that distance between points is the rectilinear distance. Each problem is to determine the location of new facilities with respect to existing facilities so as to minimize the maximum cost, where costs are specified functions of the rectilinear distance between new and existing facilities or among new facilities. For all but one of the problems considered, upper bound constraints are assumed on the rectilinear distances between new and existing facilities.

A decomposition into independent one-dimensional minimax subproblems is obtained for all but one of the problems. The optimal solutions to the subproblems determine the optimal solutions to the original problem; solution procedures are developed for the subproblems, based on their structure. The structure of some of the subproblems gives an explicit expression for their unique solution in terms of parameters of the subproblem. The structure of other subproblems enables an equivalent network flow problem to be formulated which is efficiently solved by determining a simple chain of maximum cost to weight ratio in the network.

For the remaining problem considered, properties of the rectilinear distance enable the specification of a finite iterative procedure, where each iteration solves a set covering problem, to determine an optimal location of facilities.

# CHAPTER 1

# INTRODUCTION AND OVERVIEW

## 1.1 Introduction

This dissertation contains a study of some minimax location problems in the Euclidean plane under the assumption that distance between points is the rectilinear distance. These problems may be characterized as determining the locations of new facilities with respect to existing facilities so as to minimize the maximum cost, where costs are specified functions of the rectilinear distance between new and existing facilities or among new facilities. An example of such a problem is to determine the location of one point in the plane that minimizes the maximum rectilinear distance from it to any of a finite number of fixed points in the plane.

## 1.2 Applications

Minimax location problems considered in this dissertation may have applications in determining the location of emergency service facilities such as police stations, fire stations or ambulance centers that must respond to emergencies at fixed locations. For example, it might be desired to minimize the maximum distance traveled from an ambulance center to an emergency at any fixed location. Similar applications might be found in locating emergency repair centers that must respond quickly to breakdowns at fixed facilities. Applications also may be found in locating facilities that emphasize accessibility or convenience to customers, such as public libraries, self-service post offices or banks.

The rectilinear distance assumption is made to approximate the distance between two points when travel is restricted to a system of orthogonal paths. Applications using the rectilinear distance may arise in locating facilities within a plant or warehouse where travel occurs on a system of orthogonal aisles. Applications may also be found in locating facilities in an urban area where travel is restricted to a system of orthogonal streets.

There are recognized limitations in applying the problems studied here to locate facilities in many actual settings. All the factors that require consideration for the location of some facility might not be included in the assumptions of the problems studied. However, good methodology and solution procedures for these problems form a beginning for the study of problems containing more realistic considerations. Also, solutions to some of the problems in the dissertation may be used as guidelines for locating a facility where there are additional factors to consider.

### 1.3 Related Work

Most of the problems considered in this dissertation are analogues to "total cost" location problems. The objective of these problems is to minimize the sum of costs, where costs are functions of the distance between the new and existing facilities. Perhaps the earliest total cost location problem was posed in the seventeenth century by Fermat using Euclidean distance. Since then, variations and generalizations of the problem have received considerable attention in the literature. Recent contributions to the total cost problem using Euclidean distances are reported by Kuhn [20,21], Kuhn and Kuenne [22], Love [23], and Francis and Cabot [14], and for general distances by Wendell [28]. These papers also contain extensive references to location problems. Two versions of the total cost location problem assuming rectilinear distances are considered by Francis [10,11], Wesolowsky and Love [30], and Cabot, Francis and Stary [2]. The last paper also contains an extensive list of references for location problems.

A minimax location problem using Euclidean distance was posed by Sylvester [26] in 1857. Recent contributions to two versions of this problem are reported by Elzinga and Hearn [6,7]. Properties of a minimax location problem with general distances are developed by Francis [13].

Literature concerned with minimax location problems explicitly using the rectilinear distance has appeared only since 1971. Francis [15] and Elzinga and Hearn [7] have solved a special case of one of the problems considered in this dissertation. In [16], Francis formulates special cases of other problems considered here as linear programs. Wesolowsky [29] also considers some special cases of problems considered here and suggests that they may be solved by parametric linear programming. This literature will be considered in more detail as it applies to the specific minimax problem developed in each of the following chapters.

## 1.4 Overview

Chapters 2, 3, 4 and 5 each contains a study of a different minimax location problem. Each chapter includes a formulation of the problem under consideration, a discussion of some possible applications and a review of the literature associated with the problem. Then properties of the problem are developed and a solution procedure is given.

An important aspect of the presentation in this dissertation is a transformation of the problems under consideration into equivalent minimax location problems in another space. This transformation was first applied to a minimax location problem by Francis [15]. Here the transformation is shown to apply to the more general problems that are considered. For each problem discussed in Chapters 2, 3 and 5, the transformed problem decomposes into independent subproblems which are minimax location problems in one dimension. The solutions to the subproblems provide the solutions to the original problems. For the problem studied in Chapter 4, the transformed problem does not decompose but the transformation is used in the solution procedure. Chapter 2 includes a development of this transformation.

There is also discussed in Chapter 2 the one facility minimax location problem. This problem is to locate a single facility in the plane with respect to a finite number of existing facilities. Upper bound constraints on the rectilinear distance between the new facility and the existing facilities are assumed. Properties and conditions at optimality are developed and a solution procedure based on these conditions is presented. A lower bound and a procedure for determining level sets for different values of the objective functions are also presented.

Considered in Chapter 3 is a multifacility minimax location problem for locating n new facilities with respect to existing facilities. The costs for this problem are proportional to the rectilinear distances among facilities, that is, between new and existing facilities and among new facilities. Upper bound constraints on the rectilinear distance among facilities are also assumed. The subproblems obtained from the decomposition of the transformed problem are formulated as equivalent network weighted flow problems for which a solution procedure is given.

The problem considered in Chapter 4 is a multifacility minimax location problem referred to as the p-center problem. This problem is to locate p new facilities with respect to existing facilities so as to minimize the maximum cost. In the p-center problem, there is a cost for each existing facility which is the minimum, over the new facilities, of a linear function of the rectilinear distance between the existing facility and each new facility. From the transformed problem there is determined a sequence of set covering problems. The solutions to these set covering problems determine the optimal p-center for all possible values of p.

Discussed in Chapter 5 is a one facility minimax location problem for locating a single facility with respect to existing facilities, but where the existing facilities are points of a compact subset of the plane. Properties developed by Francis [13] for minimax location problems with respect to a finite set of existing facilities are extended to a compact set of existing facilities. These properties together with the results of Chapter 2 are used to develop properties and solution procedures for the problem.

Chapter 6 contains a summary of the dissertation and suggests some areas for further research.

# CHAPTER 2

# A ONE FACILITY PROBLEM

## 2.1 Introduction

The problem considered in this chapter is to locate one new facility with respect to a finite number of existing facilities so as to minimize the maximum cost, where costs are linear functions of the rectilinear distance between the new and existing facilities. The problem also includes a constraint for each existing facility which is an upper bound on the rectilinear distance from that facility to the new facility.

A transformation, introduced in [15], is shown to apply to this problem and to give a decomposition into two independent one-dimensional minimax subproblems. All optimal solutions to the original one facility problem are obtained from the solutions to the subproblems. Solution procedures are developed for each subproblem in terms of the maximum of a finite number of simple functions of the subproblem's parameters.

## 2.2 Formulation

The one facility minimax location problem is now formulated and notation to be used throughout the dissertation is defined.

A notational convenience adopted here is that "$1 \leq i \leq m$" denotes all the integers and only the integers from 1 to m.

Let $X = (x,y)$ and $P = (a,b)$ by any two points in the Euclidean plane, $E_2$. Denote the rectilinear distance between $X$ and $P$ as $\|X - P\|_1$ so that

$$\|X - P\|_1 = \|(x,y) - (a,b)\|_1 = |x - a| + |y - b|.$$

Let $X$ denote the point in $E_2$ to be determined for the location of the new facility. Let $S$ be a finite set in $E_2$ of $m$ points $P_i$, $1 \leq i \leq m$, representing existing facilities. For each $P_i$ let $g_i$ be any real constant and let $w_i$ be any real positive constant.

Define a function $f(X)$ by

$$f(X) = \max_{1 \leq i \leq m} \left\{ w_i \|X - P_i\|_1 + g_i \right\}.$$

Each expression, $w_i \|X - P_i\|_1 + g_i$, for $1 \leq i \leq m$, is a linear function of the rectilinear distance between $X$ and $P_i$, referred to previously as a cost. Results in [12] extend to show that each of these expressions is a continuous, convex function in $X$. Thus the function $f(X)$ is continuous and convex since it is the pointwise maximum of a finite number of continuous, convex functions in $X$.

For each point $P_i$ let $c_i$ be a positive constant for $1 \leq i \leq m$ which is an upper bound on the rectilinear distance between $P_i$ and $X$.

With the above terminology, the one facility minimax location problem to be studied in this chapter can be stated as

$$\begin{aligned} &\min_{X \in E_2} \max_{1 \leq i \leq m} \left\{ w_i \|X - P_i\|_1 + g_i \right\} \\ &\text{s.t.} \quad \|X - P_i\|_1 \leq c_i \quad 1 \leq i \leq m. \end{aligned} \tag{2.2.1}$$

Observe that the unconstrained one facility minimax location problem

$$\min_{X \epsilon E_2} \max_{1 \leq i \leq m} \left\{ w_i \| X - P_i \|_1 + g_i \right\} \tag{2.2.2}$$

is a special case of problem (2.2.1). This follows by taking the $c_i$ in problem (2.2.1) sufficiently large so that the constraints are not binding.

A generalization of problem (2.2.1) considered in Section 2.8 allows the $w_i$ to be nonnegative. However, if all $w_i = 0$, the objective function value is independent of the location of X, and the problem is not a well-formulated location problem. Therefore, it will be assumed that some of the $w_i$ are positive when considering this generalization.

## 2.3 Applications

Possible applications for the one facility minimax location problem are included in those uses suggested in Chapter 1 for more general minimax location problems.

As a specific example, [16,19], suppose it is desired to locate an ambulance center in a city to respond to m existing facilities in the city. Each response involves traveling to one of the existing facilities and then to a hospital. The ambulance center is to be located so that the maximum distance traveled to respond to an emergency is minimized. In addition, there are requirements that the ambulance center be within a specified distance from each existing facility.

Let $P_i$, for $1 \leq i \leq m$, be the location of an existing facility. Take $w_i = 1$ and $g_i$ to be the distance from the facility at $P_i$ to the hospital. Take $c_i$ to be the upper bound on the distance that the ambulance center must be located from the facility at $P_i$. Then this problem may be formulated as problem (2.2.1).

The numbers $w_i$ may have several interpretations as constants per unit of distance between X and $P_i$ with corresponding interpretations for the constants $g_i$. For example, in locating a service repair center at some point X in a plant, it might be desired to minimize the maximum cost of travel from X to the fixed locations $P_i$. In this application the $w_i$ may be interpreted as costs per unit distance and the $g_i$ as constant costs. In another application of locating an emergency center at some point X, it might be desired to minimize the maximum time to travel from the fixed locations $P_i$ to X. Then $w_i$ would be interpreted as time per unit distance and the $g_i$ as fixed times associated with each point $P_i$.

The $w_i$ may also be interpreted as a weight or a measure of relative importance given to each point $P_i$. For example, in locating a bank in a city, customers located at the points $P_i$ may be given different weights according to their frequency of travel to, or level of business with, the bank. Then locating the bank so as to minimize the maximum weighted distance that customers have to travel may be investigated using the problem formulated as (2.2.1).

Another interpretation of the points $P_i$ is obtained by idealizing an area as a point. Then the one facility problem may find applications in locating a facility with respect to given areas or regions in an urban or industrial setting.

## 2.4 Related Work

Francis [12] considered properties of a one facility minimax location problem for general distances in 1966. However, the one facility problem using the rectilinear distance appears to have received explicit consideration only since 1971. Elzinga and Hearn [7] consider a geometrically based procedure that gives an explicit solution to a special case of problem (2.2.2) obtained by setting $w_i = 1$ and $g_i \geq 0$ for $1 \leq i \leq m$. Francis [15] considers this problem with $g_i = 0$, and introduces a transformation of the problem that gives an explicit solution procedure. This transformation is developed in Section 2.5 for application to problem (2.2.1) and to the other minimax location problems considered in the sequel.

In [16], Francis considers some special cases of problem (2.2.2) and formulates them as equivalent linear programs. For the special case with $w_i = 1$ and $g_i = 0$, he obtains an explicit solution based on the structure of the linear program.

Using the development in [16], an equivalent linear program to (2.2.1) is obtained. A new variable z is introduced and the constrained problem

$$\begin{aligned} &\min z \\ &\text{s.t.} \quad \left.\begin{aligned} w_i \|X - P_i\|_1 + g_i &\leq z \\ \|X - P_i\|_1 &\leq c_i \end{aligned}\right\} 1 \leq i \leq m \end{aligned} \tag{2.4.1}$$

is written which is equivalent to problem (2.2.1). Each constraint of (2.4.1) may be written equivalently as four linear inequality constraints, giving a linear programming problem. The linear programming problem is not used directly in the solution procedure developed in the sequel; however, this formulation is useful in proving some results that follow and in providing insight into the one facility problem.

Wesolowsky [29] considers the problem (2.2.2) with $w_i > 0$ and $g_i = 0$, $1 \leq i \leq m$. He suggests a graphical solution procedure but notes "when m is large this graphical trial and error method is unwieldy." He also reports that the problem can be written as a linear program but erroneously states "the approach is only workable in the one-dimensional case."

Thus, while the literature reports that some special cases of problem (2.2.1) are equivalent to linear programs, an explicit solution procedure for the problem has not been developed. The notion of the transformation does not appear to have been applied to minimax location problems previous to [15] and has not been developed as a general tool for application to the minimax location problems studied here.

## 2.5 A Transformation and Decomposition

This section provides an analytical development of a transformation that is applied to the one facility problem and allows a decomposition into simpler subproblems. A geometrical development is given for this transformation in [15] where it is applied to a special case of problem (2.2.2). The geometrical development is based on the observation that the locus of all points in $E_2$ of equal rectilinear distance from a point X in $E_2$ is a "diamond" with sides of equal length and slope ± 1. Rotating this diamond by 45 degrees gives a square with sides parallel to the axes, and a square is the locus of all points in $E_2$ of equal Chebyshev distance from a point X in $E_2$.

The Chebyshev distance between two points X and P in $E_2$, denoted by $\| X - P \|_\infty$, is defined by

$$\| X - P \|_\infty = \| (x,y) - (a,b) \|_\infty = \max \left( |x-a|, \ |y-b| \right) .$$

The transformation, denoted by T, is a counterclockwise rotation of the axes through 45 degrees and is given by the nonsingular matrix

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .$$

The first property relates the rectilinear distance in $E_2$ to the Chebyshev distance in $T(E_2)$. This result is stated without proof by Beckenbach and Bellman [1] and by Francis [15], but it is given and proved here for completeness.

Property 2.5.1. $\| X - P \|_1 = \sqrt{2} \, \| T(X) - T(P) \|_\infty$ for any two points X and P in $E_2$.

Proof: $\sqrt{2} \, \| T(x,y) - T(a,b) \|_\infty = \sqrt{2} \max \left\{ 1/\sqrt{2} |x+y-a-b| , \ 1/\sqrt{2} |-x+y+a-b| \right\}$

$$= \max \left\{ (x+y-a-b), \ (-x-y+a+b), \ (-x+y+a-b), \ (x-y-a+b) \right\}$$

$$= \max \left\{ (x-a+y-b), \ (x-a-y+b), \ (-x+a+y-b), \ (-x+a-y+b) \right\}$$

$$= |x - a| + |y - b|$$

$$= \|X - P\|_1 .$$

Now let $P' = T(P)$ be the image of a point P under T. Define

$$S' = T(S) = \left\{P'_i : P'_i = T(P_i),\ P_i \in S\right\} .$$

Let $g'_i = g_i/\sqrt{2}$ and $c'_i = c_i/\sqrt{2}$ for $1 \leq i \leq m$. Consider the problem

$$\begin{aligned} &\min_{X' \in E_2} \max_{1 \leq i \leq m} \left\{w_i \|X' - P'_i\|_\infty + g'_i\right\} \\ &\text{s.t.} \quad \|X' - P'_i\|_\infty \leq c'_i \qquad 1 \leq i \leq m. \end{aligned} \tag{2.5.1}$$

Problem (2.5.1) will be referred to as the rotated one facility problem.

Using Property 2.5.1, the equivalence of problems (2.2.1) and (2.5.1) is established in the next property.

Property 2.5.2. $X_o$ is an optimal solution to problem (2.2.1) if and only if $T(X_o)$ is an optimal solution to problem (2.5.1). Furthermore, if $z_o$ and $z'_o$ are the optimal objective function values for problems (2.2.1) and (2.5.1), respectively, then

$$z_o = \sqrt{2}\, z'_o . \tag{2.5.2}$$

Proof: The proof is an application of Property 2.5.1.

Writing problem (2.5.1) using the definition of the Chebyshev distance gives

$$\min_{X' \epsilon E_2} \max_{1 \leq i \leq m} \left\{ w_i \left[ \max \left( |x' - a'_i|, |y' - b'_i| \right) \right] + g'_i \right\}$$

$$\text{s.t.} \quad \max \left( |x' - a'_i|, |y' - b'_i| \right) \leq c'_i \quad 1 \leq i \leq m .$$

Each constraint of the above problem may be written as two constraints and the order of the maximizations may be interchanged in the objective function to give

$$\min_{X' \epsilon E_2} \max \left[ \max_{1 \leq i \leq m} \left\{ w_i |x' - a'_i| + g'_i \right\} , \max_{1 \leq i \leq m} \left\{ w_i |y' - b'_i| + g'_i \right\} \right]$$

$$\text{s.t.} \quad \begin{array}{l} |x' - a'_i| \leq c'_i \\ \\ |y' - b'_i| \leq c'_i \end{array} \quad 1 \leq i \leq m . \tag{2.5.3}$$

From the above problem, two subproblems are defined.

$$\min_{x' \epsilon E_1} \max_{1 \leq i \leq m} \left\{ w_i |x' - a'_i| + g'_i \right\}$$

$$\text{s.t.} \quad |x' - a'_i| \leq c'_i \quad 1 \leq i \leq m \tag{2.5.4}$$

and

$$\min_{y' \epsilon E_1} \max_{1 \leq i \leq m} \left\{ w_i |y' - b'_i| + g'_i \right\}$$

$$\text{s.t.} \quad |y' - b'_i| \leq c'_i \quad 1 \leq i \leq m. \tag{2.5.5}$$

Each subproblem is a one facility minimax location problem in one dimension.

The following property relates the optimal solutions of the subproblems to the rotated problem, (2.5.1).

Property 2.5.3. If $x'_o$ is an optimal solution to subproblem (2.5.4) with objective function value $z'_1$ and if $y'_o$ is an optimal solution to subproblem (2.5.5) with objective function value $z'_2$, then $(x'_o, y'_o)$ is an optimal solution to problem (2.5.1) with optimal objective function value $z'_o = \max (z'_1, z'_2)$.

Properties 2.5.2 and 2.5.3 show that the optimal solutions to the one facility problem (2.2.1) are obtained from the independent solutions of the subproblems (2.5.4) and (2.5.5). In the next section some properties of the subproblems are developed.

Notice that the transformation developed in this section for problem (2.2.1) in $E_2$ does not extend directly to analogous problems in $E_3$. The locus of all points of equal rectilinear distance from a point X in $E_3$ is the surface of an octahedron; whereas the locus of all points of equal Chebyshev distance from a point X in $E_3$ is the surface of a cube. Neither of these figures is the image of the other under a linear transformation.

## 2.6 Subproblem Development

This section develops properties of the subproblems that form a basis for their solution.

Each subproblem is first rewritten with simplified constraints. Each constraint of problem (2.5.4) may be written equivalently as two inequality constraints: $x' \geq a'_i - c'_i$ and $x' \leq a'_i + c'_i$ for $1 \leq i \leq m$.

Let

$$C_1' = \max_{1 \le i \le m} (a_i' - c_i')$$

$$C_2' = \min_{1 \le i \le m} (a_i' + c_i').$$

Then problem (2.5.4) is equivalent to the problem

$$\begin{aligned} &\min_{x' \in E_1} \max_{1 \le i \le m} \left\{ w_i \left| x' - a_i' \right| + g_i' \right\} \\ &\text{s.t.} \quad C_1' \le x' \le C_2' \,. \end{aligned} \tag{2.6.1}$$

Similarly for problem (2.5.5), let

$$D_1' = \max_{1 \le i \le m} (b_i' - c_i')$$

$$D_2' = \min_{1 \le i \le m} (b_i' + c_i')$$

and problem (2.5.5) is equivalent to

$$\begin{aligned} &\min_{y' \in E_1} \max_{1 \le i \le m} \left\{ w_i \left| y' - b_i' \right| + g_i' \right\} \\ &\text{s.t.} \quad D_1' \le y' \le D_2' \,. \end{aligned} \tag{2.6.2}$$

The subproblems (2.6.1) and (2.6.2) provide a test for the feasibility of the original problem (2.2.1). That is, problem (2.2.1) has a feasible solution if and only if each subproblem, (2.6.1) and (2.6.2), has a feasible solution. Either subproblem (2.6.1) or (2.6.2) is infeasible if $C_2 < C_1$, or $D_2 < D_1$, respectively.

Problems (2.6.1) and (2.6.2) have the same structure so that for any property of (2.6.1) there is an analogous property for (2.6.2). Thus it is sufficient to develop the following properties only for problem (2.6.1).

Problem (2.6.1) without constraints is written as

$$\min_{x' \epsilon E_1} \max_{1 \leq i \leq m} \left\{ w_i \left| x' - a'_i \right| + g'_i \right\} . \qquad (2.6.3)$$

If $x'_o$ is an optimal solution to problem (2.6.3) and $x'_o$ is feasible to problem (2.6.1), then $x'_o$ is an optimal solution to (2.6.1). The case where $x'_o$ is not feasible to problem (2.6.1) is considered by the next property.

Property 2.6.1. If $x'_o$ is an optimal solution to (2.6.3) and $x'_o$ is infeasible to (2.6.1), then the optimal solution to (2.6.1) is either $C'_1$ or $C'_2$. Specifically, if $x'_o < C'_1$, $C'_1$ is the unique optimal solution to (2.6.1), or if $x'_o > C'_2$, $C'_2$ is the unique optimal solution to (2.6.1).

The proof of Property 2.6.1 is given after the proof of Property 2.6.2.

Property 2.6.1 shows that it is sufficient to obtain solutions only to the unconstrained problem, (2.6.3), when solving problem (2.6.1). The following properties are developed for the unconstrained problem.

Property 2.6.2. Problem (2.6.3) has a unique optimal solution.

Proof: The linear programming problem for (2.6.3) is

$$\begin{aligned} &\min z' \\ &\text{s.t.} \quad x' + z'/w_i \geq a'_i + g'_i/w_i \\ &\qquad -x' + z'/w_i \geq -a'_i + g'_i/w_i \qquad 1 \leq i \leq m \qquad (2.6.4) \\ &\qquad x', z' \text{ unrestricted.} \end{aligned}$$

Suppose that with optimal objective function value $z'_1$, problem (2.6.3) has alternative optima $x'_1$ and $x'_2$ with $x'_1 \neq x'_2$. Then, $x'_1$ and $x'_2$ must satisfy some common constraint, that is, both points lie on some edge of the feasible set. Then, either there exists an i such that

$$x'_1 + z'_1/w_i = a'_i + g'_i/w_i \quad \text{and} \quad x'_2 + z'_1/w_i = a'_i + g'_i/w_i$$

or there exists a j such that

$$-x'_1 + z'_1/w_j = -a'_j + g'_j/w_j \quad \text{and} \quad -x'_2 + z'_1/w_j = -a'_j + g'_j/w_j \ .$$

In either case, eliminating $z'_1$ yields $x'_1 = x'_2$, a contradiction.

Proof of Property 2.6.1: Write the objective function of problem (2.6.3) as

$$f_1(x') = \max_{1 \le i \le m} \left\{ w_i \left| x' - a'_i \right| + g'_i \right\} .$$

Then, $f_1(x')$ is a continuous, convex function in $x'$ with a unique minimum, say $x'_o$. These properties imply that $f_1(x')$ is strictly increasing in the interval $[x'_o, \infty)$ and strictly decreasing in the interval $(-\infty, x_o]$. Then, if $x'_o < C'_1 < x' < C'_2$, $f_1(C'_1) < f_1(x')$ for all $x'$ in $(C'_1, C'_2)$. If $C'_1 < x' < C'_2 < x'_o$, then $f_1(x') > f_1(C'_2)$ for all $x'$ in $(C'_1, C'_2)$.

Next a lower bound is given for problem (2.6.3). A bound was obtained by Francis [12] for problem (2.6.3) with $g'_i = 0$ and is extended here to the problem with $g'_i \neq 0$. For each i and j with $1 \le i \le j \le m$, define the number

$$b_{ij} = \max \left\{ \frac{1}{w_i + w_j} \left[ w_i w_j \left| a'_i - a'_j \right| + w_j g'_i + w_i g'_j \right], \ g'_i, \ g'_j \right\}$$

and define

$$b_{st} = \max_{1\le i<j\le m} \left\{b_{ij}\right\} . \qquad (2.6.5)$$

Property 2.6.3. $b_{st} \le \min_{x'\varepsilon E_1} \max_{1\le i\le m} \left\{w_i \left|x' - a_i'\right| + g_i'\right\}$ .

The proof of Property 2.6.3 is a direct extension of that given in [12] and is omitted. The next property is a new result and establishes that this lower bound is always attained at optimality for problem (2.6.3).

Property 2.6.4. The optimal objective function value of problem (2.6.3) equals its respective lower bound.

Proof: The dual to the linear program, (2.6.4), is

$$\max\ (a_1'+g_1'/w_1)u_1 + .. + (a_m'+g_m'/w_m)u_m + (-a_1'+g_1'/w_1)v_1 + .. + (-a_m'+g_m'/w_m)v_m$$

$$\begin{aligned} \text{s.t.}\quad &(1)\quad u_1 + u_2 + \ldots + u_m - v_1 - v_2 - \ldots - v_m = 0 \\ &(2)\quad u_1/w_1 + \ldots + u_m/w_m + v_1/w_1 + . . + v_m/w_m = 1 \\ &(3)\quad u_i,\ v_i \ge 0 \qquad\qquad 1 \le i \le m. \end{aligned}$$

The constraints imply that any basis matrix to the dual must be of the form

$$\begin{pmatrix} 1 & -1 \\ 1/w_i & 1/w_j \end{pmatrix}$$

for some i and j. Constraints 1 and 3 imply that $u_i = v_j$ so that constraint 2 gives

$$u_i/w_i + v_j/w_j = u_i(1/w_i + 1/w_j) = 1$$

and

$$u_i = \frac{w_i w_j}{w_i + w_j} .$$

Thus the optimal objective function value has the form

$$\max_{1 \le i \le j \le m} \left\{ u_i(a'_i + g'_i/w_i - a'_j + g'_j/w_j) \right\}$$

$$= \max_{1 \le i < j \le m} \left\{ \frac{1}{w_i + w_j} \left[ w_i w_j (a'_i - a'_j) + g'_i w_j + g'_j w_i \right] , g'_i , g'_j \right\}$$

which is the lower bound.

The next property is an extension of a similar result in [12].

Property 2.6.5. Suppose $b_{st}$ is the lower bound for (2.6.3), determined by $a'_s$ and $a'_t$. Define

$$x_{st} = \frac{1}{w_s + w_t} \left[ w_s a'_s + w_t a'_t + g'_t - g'_s \right] . \tag{2.6.6}$$

Then $x_{st}$ is the optimal solution to (2.6.3).

Proof: For $a'_s < a'_t$, consider the basis matrix B for the dual problem and its inverse $B^{-1}$.

$$B = \begin{pmatrix} 1 & -1 \\ 1/w_t & 1/w_s \end{pmatrix} \qquad B^{-1} = \frac{w_s w_t}{w_s + w_t} \begin{pmatrix} 1/w_s & 1 \\ -1/w_t & 1 \end{pmatrix} .$$

Then the optimal solution to the primal is given by

$$(a'_t + g'_t/w_t , \; - a'_s + g'_s/w_s) B^{-1} = (x'_o , z'_1)$$

which gives

$$z_1' = \frac{1}{w_s+w_t}\left[w_s w_t(a_t' - a_s') + g_t'w_s + g_s'w_t\right] = b_{st}$$

$$x_o' = \frac{1}{w_s+w_t}\left[w_t a_t' + w_s a_s' + g_t' - g_s'\right].$$

Since the optimal solution is unique, it is given by (2.6.6). Note that if $a_s' = a_t'$, equation (2.6.6) gives $x_{st} = a_s' = a_t'$, and in this case the bound is given by the max $(g_s', g_t') = b_{ss}$ or $b_{tt}$. Then for the case $b_{ss}$, for example, the dual basis gives $x_o' = a_s'$ and $z_1' = g_s'$.

## 2.7 Solution Procedure

The properties developed in the preceding section for the subproblem (2.6.1) and the analogous properties that exist for subproblem (2.6.2) are used to determine a solution procedure for both subproblems. Using the optimal solutions to the subproblems, a procedure is given for finding all alternative solutions to the one facility problem in two dimensions, (2.2.1).

Subproblem (2.6.1) is solved by computing the lower bound, (2.6.5), and determining the value $x_o'$ using (2.6.6). If $x_o'$ is feasible to problem (2.6.1), then the unique optimal solution is $x_o' = x_{st}'$, and $z_1' = b_{st}$. If it is not feasible, the unique optimal solution is $x_o' = C_1'$ or $x_o' = C_2'$ as determined by Property 2.6.1, and

$$z_1' = \max_{1 \leq i \leq m} \left\{w_i \left|C_j' - a_i'\right| + g_i'\right\}$$

for j = 1 or 2, respectively.

Similarly, subproblem (2.6.2) is solved by computing the lower bound

$$b_{st} = \max_{1 \leq i < j \leq m} \left\{ \frac{1}{w_i + w_j} \left[ w_i w_j |b'_i - b'_j| + w_j g'_i + w_i g'_j \right],\ g'_i, g'_j \right\} .$$

Then compute $y_{st}$ using

$$y'_{st} = \frac{1}{w_s + w_t} \left[ w_s b'_s + w_t b'_t + g'_t - g'_s \right] .$$

If $y'_{st}$ is feasible to (2.6.2), the unique optimal solution to (2.6.2) is $y'_o = y'_{st}$ and $z'_1 = b_{st}$; if not, the unique optimal solution is $y'_o = D'_1$ or $y'_o = D'_2$ and

$$z'_2 = \max_{1 \leq i \leq m} \left\{ w_i |D'_j - b'_i| + g'_i \right\}$$

for j = 1 or 2, respectively.

By Property 2.5.3, the optimal objective function value, $z'_o$, to the rotated problem, (2.5.1), is given by $z'_o = \max (z'_1, z'_2)$ where $z'_1$ and $z'_2$ are the optimal objective function values to (2.6.1) and (2.6.2), respectively.

The point $(x'_o, y'_o)$ is one optimal solution to (2.5.1) by Property 2.5.3. Three cases need to be considered to determine all the alternative solutions.

Case 1: $z'_o = z'_1 = z'_2$. Then $(x'_o, y'_o)$ is the unique optimal solution to (2.5.1).

Case 2: $z'_o = z'_1 > z'_2$. Then for all y' that satisfy the inequalities

$$w_i |y' - b'_i| + g'_i \le z'_o \qquad 1 \le i \le m$$

and (2.7.1)

$$D'_1 \le y' \le D'_2$$

the points $(x'_o, y')$ are also optimal to (2.5.1). For $1 \le i \le m$, the $i^{th}$ inequality in (2.7.1) may be written as two linear inequalities so that (2.7.1) is equivalent to

$$b'_i - (z'_o - g'_i)/w_i \le y' \le b'_i + (z'_o - g'_i)/w_i \qquad 1 \le i \le m$$

$$D'_1 \le y' \le D'_2.$$

These inequalities determine an interval, $(y'_1, y'_2)$, of alternative $y'$ where

$$y'_1 = \max\left[D'_1, \max_{1 \le i \le m} \left\{b'_i - (z'_o - g'_i)/w_i\right\}\right]$$

(2.7.2)

$$y'_2 = \min\left[D'_2, \min_{1 \le i \le m} \left\{b'_i + (z'_o - g'_i)/w_i\right\}\right].$$

Then the points $(x'_o, y')$ on the line segment determined by $(x'_o, y'_1)$ and $(x'_o, y'_2)$ are the optimal solutions to (2.5.1).

Case 3: $z'_o = z'_2 > z'_1$. Then determine

$$x'_1 = \max\left[C'_1, \max_{1 \le i \le m} \left\{a'_i - (z'_o - g'_i)/w_i\right\}\right]$$

(2.7.3)

$$x'_2 = \min\left[C'_2, \min_{1 \le i \le m} \left\{a'_i + (z'_o - g'_i)/w_i\right\}\right].$$

Then the points $(x', y'_o)$ on the line segment determined by $(x'_1, y'_o)$ and $(x'_2, y'_o)$ are all optimal solutions to problem (2.5.1).

By Property 2.5.2 the optimal solution to the problem (2.2.1) is obtained by taking the inverse rotation, $T^{-1}$, of the solutions to (2.5.1). In Case 1, $(x_o, y_o) = T^{-1}(x'_o, y'_o)$ is the unique solution to (2.2.1). For Cases 2 and 3 with a line segment of solutions, the image of a line segment under a nonsingular transformation of $E_2$ onto $E_2$ is a line segment. Thus for Case 2, the points $(x_o, y_o)$ on the line segment determined by $(x_1, y_1) = T^{-1}(x'_o, y'_1)$ and $(x_2, y_2) = T^{-1}(x'_o, y'_2)$ are all optimal solutions to (2.2.1). Similarly for Case 3, the optimal solutions are on the line segment determined by the points $(x_1, y_1) = T^{-1}(x'_1, y'_o)$ and $(x_2, y_2) = T^{-1}(x'_2, y'_o)$.

## 2.8 Additional Properties

Using the above developments, additional properties of the one facility problem are given.

A lower bound is determined for the problem (2.2.1) and it is shown that for the unconstrained problem, (2.2.2), this lower bound is always attained at optimality. A similar bound is obtained in [12] and is extended here to the case with $g_i \neq 0$. Define the numbers $B_{ij}$ for each i and j such that $1 \leq i \leq j \leq m$ by

$$B_{ij} = \max \left\{ \frac{1}{w_i + w_j} \left[ w_i w_j \| P_i - P_j \|_1 + w_j g_i + w_i g_j \right], g_i, g_j \right\}$$

and define

$$B_{st} = \max_{1 \leq i < j \leq m} \left\{ B_{ij} \right\} .$$

Property 2.8.1. If $z_o$ is the optimal objective function value of problem (2.2.1), then $B_{st} \leq z_o$.

Proof: By a direct extension of the proof given [12], for any X in $E_2$,

$$B_{st} \leq \max_{1 \leq i \leq m} \left\{ w_i \| X - P_i \|_1 + g_i \right\} . \qquad (2.8.1)$$

Therefore inequality (2.8.1) holds for all feasible points, X, to problem (2.2.1) and for the minimum point, $X_o$, in the feasible region. Hence,

$$B_{st} \leq \max_{1 \leq i \leq m} \left\{ w_i \| X_o - P_i \|_1 + g_i \right\} = z_o .$$

Property 2.8.2. The optimal objective function value of the unconstrained problem, (2.2.2), equals the lower bound $B_{st}$.

Proof: Let $z'_o$ be the optimal objective function value of the rotated problem, (2.5.1), without constraints. Then by Property 2.5.3,

$$z'_o = \max \left\{ z'_1 , z'_2 \right\}$$

where $z'_1$ and $z'_2$ are the respective lower bounds of the subproblems (2.6.1) and (2.6.2) without constraints. Then

$$z'_o = \max \left\{ \max_{1 \leq i < j \leq m} \left\{ \frac{1}{w_i + w_j} \left[ w_i w_j | a'_i - a'_j | + w_i g'_j + w_j g'_i \right] , g'_i , g'_j \right\} , \right.$$

$$\left. \max_{1 \leq i < j \leq m} \left\{ \frac{1}{w_i + w_j} \left[ w_i w_j | b'_i - b'_j | + w_i g'_j + w_j g'_i \right] , g'_i , g'_j \right\} \right\}$$

Property 2.5.2 implies that $z_o' = z_o/\sqrt{2}$. Applying the transformation $T^{-1}$, where $a_i' = (a_i + b_i)/\sqrt{2}$ and $b_i' = (-a_i + b_i)/\sqrt{2}$, and multiplying through by $\sqrt{2}$ gives

$$
\begin{aligned}
z_o = \max \Bigg\{ & \max_{1 \leq i < j \leq m} \left\{ \frac{1}{w_i + w_j} \left[ w_i w_j \left| a_i + b_i - a_j - b_j \right| + w_i g_j + w_j g_i \right], g_i, g_j \right\}, \\
& \max_{1 \leq i < j \leq m} \left\{ \frac{1}{w_i + w_j} \left[ w_i w_j \left| -a_i + b_i + a_j - b_j \right| + w_i g_j + w_j g_i \right], g_i, g_j \right\} \Bigg\} \\
= & \max_{1 \leq i < j \leq m} \Bigg\{ \frac{1}{w_i + w_j} \Bigg[ w_i w_j \max \Big( \left| a_i + b_i - a_j - b_j \right|, \\
& \qquad \left| -a_i + b_i + a_j - b_j \right| \Big) + w_i g_j + w_j g_i \Bigg], g_i, g_j \Bigg\} \\
= & \max_{1 \leq i < j \leq m} \left\{ \frac{1}{w_i + w_j} \left[ w_i w_j \left( \left| a_i - a_j \right| + \left| b_i - b_j \right| \right) + w_i g_j + w_j g_i \right], g_i, g_j \right\} \\
= & \max_{1 \leq i < j \leq m} \left\{ \frac{1}{w_i + w_j} \left[ w_i w_j \left\| P_i - P_j \right\|_1 + w_i g_j + w_j g_i \right], g_i, g_j \right\} \\
= & \max_{1 \leq i < j \leq m} \left\{ B_{ij} \right\} .
\end{aligned}
$$

The above results are now used to develop a procedure to determine level sets and contour sets for problem (2.2.1).

For the function f(X) and for a real positive number k, a level set with respect to k, denoted by $S_k$, is defined by

$$S_k = \left\{ X \in E_2 : f(X) \leq k \right\} .$$

A contour set, for the function $f(X)$, with respect to $k$ is denoted by $T_k$, and is defined by

$$T_k = \left\{ X \in E_2 : f(X) = k \right\} .$$

Denote the set of feasible points to problem (2.2.1) by F. Then a level set with respect to k of problem (2.2.1) is $F \cap S_k$ and a contour set of problem (2.2.1) with respect to k is $F \cap T_k$.

The level set, $F \cap S_k$, is the set of points X such that

$$w_i \| X - P_i \|_1 + g_i \leq k \qquad 1 \leq i \leq m \tag{2.8.2}$$

$$\| X - P_i \|_1 \leq c_i \qquad 1 \leq i \leq m. \tag{2.8.3}$$

The points X satisfying (2.8.2) and (2.8.3) may be determined using the preceding development. That is, transform the inequalities by T, decompose the resulting inequalities and determine the points $(x_1', y_1')$ $(x_1', y_2')$ $(x_2', y_1')$ $(x_2', y_2')$, using expressions (2.7.2) and 2.7.3) with k replacing $z_o'$. Then the corners of the set of solutions are determined by taking the inverse transformation of these points.

Alternatively, the points X in the level set $F \cap S_k$ may be determined directly from the expressions (2.8.2) and (2.8.3). For each i, the $i^{th}$ inequality in (2.8.2) and (2.8.3) is equivalent to four linear inequalities. Define $G_i(k) = (k - g_i)/w_i$. Then the equivalent linear inequalities are

$$\left.\begin{array}{r} x + y \geq a_i + b_i - G_i(k) \\ x - y \geq a_i - b_i - G_i(k) \\ -x + y \geq -a_i + b_i - G_i(k) \\ -x - y \geq -a_i - b_i - G_i(k) \end{array}\right. \qquad 1 \leq i \leq m$$

and

$$\left.\begin{aligned} x + y &\geq a_i + b_i - c_i \\ x - y &\geq a_i - b_i - c_i \\ -x + y &\geq -a_i + b_i - c_i \\ -x - y &\geq -a_i - b_i - c_i . \end{aligned}\right. \qquad 1 \leq i \leq m$$

All but four inequalities from the above are redundant. To eliminate the redundant inequalities define the numbers, dependent on k,

$$e_1(k) = \max\left[\max_{1\leq i\leq m} (a_i + b_i - G_i(k)),\ \max_{1\leq i\leq m} (a_i + b_i - c_i)\right]$$

$$e_2(k) = \max\left[\max_{1\leq i\leq m} (a_i - b_i - G_i(k)),\ \max_{1\leq i\leq m} (a_i - b_i - c_i)\right]$$

$$(2.8.4)$$

$$e_3(k) = \max\left[\max_{1\leq i\leq m} (-a_i + b_i - G_i(k)),\ \max_{1\leq i\leq m} (-a + b_i - c_i\right]$$

$$e_4(k) = \max\left[\max_{1\leq i\leq m} (-a_i - b_i - G_i(k)),\ \max_{1\leq i\leq m} (-a_i - b_i - c_i)\right].$$

Then the four inequalities

$$\begin{aligned} x + y &\geq e_1(k) \\ x - y &\geq e_2(k) \\ -x + y &\geq e_3(k) \\ -x - y &\geq e_4(k) \end{aligned} \qquad (2.8.5)$$

are equivalent to the inequalities (2.8.2) and (2.8.3), and the set of points $X = (x,y)$ satisfying (2.8.5) are the points in the level set $F \cap S_k$.

It can be shown that the contour set, $F \cap T_k$, is the boundary of the level set $F \cap S_k$. Hence to determine the contour set, use the expressions (2.8.4) to determine $e_1(k)$, $e_2(k)$, $e_3(k)$ and $e_4(k)$. Then write the expressions (2.8.5) as equalities, as follows:

$$\begin{aligned} x + y &= e_1(k) \\ x - y &= e_2(k) \\ -x + y &= e_3(k) \\ -x - y &= e_4(k) \ . \end{aligned}$$

Then the set of all $X = (x,y)$ satisfying these equations are the points in the contour set $F \cap T_k$. Notice that if k is chosen as the optimal objective function value, $F \cap T_k$ is the set of optimal solutions to problem (2.2.1).

In the previous development it was assumed that $w_i > 0$ for all $P_i$ in S. The following procedure considers the case where $w_i = 0$ for some $P_i$ in S. Let

$$W = \left\{ i : P_i \in S,\ w_i > 0 \right\} .$$

Then the problem (2.2.1) may be written as

$$\min_{X \in E_2} \ \max \left[ \max_{i \in W} \left\{ w_i \| X - P_i \|_1 + g_i \right\} , \max_{\substack{1 \le i \le m \\ i \notin W}} \left\{ g_i \right\} \right]$$

$$\text{s.t.} \qquad \| X - P_i \|_1 \le c_i \qquad 1 \le i \le m \ .$$

The subproblem

$$\min_{X \in E_2} \ \max_{i \in W} \left\{ w_i \| X - P_i \|_1 + g_i \right\}$$

$$\text{s.t.} \qquad \| X - P_i \|_1 \le c_i \qquad 1 \le i \le m$$

may be solved, using the procedure determined above, for its optimal objective function value $z_o$. Let

$$g_o = \max_{\substack{1 \leq i \leq m \\ i \notin W}} \{g_i\} .$$

If $z_o \geq g_o$, then the solution to problem (2.2.1) is given by the solution to the subproblem over the set W. If $z_o < g_o$, than all X such that

$$w_i \| X - P_i \|_1 + g_i \leq g_o \quad \text{for all} \quad i \text{ in } W \tag{2.8.6}$$

and

$$\| X - P_i \|_1 \leq c_i \quad \text{for all} \quad 1 \leq i \leq m \tag{2.8.7}$$

are solutions to the original problem with optimal objective function value $g_o$.

Because of the similarity between (2.8.6), (2.8.7) and (2.8.2), (2.8.3), the solutions may be determined using the procedure developed for the level sets. That is, in (2.8.4) set $g_o = k$ and take the maximum of the first terms over the index set W to determine $e_1(g_o)$, $e_2(g_o)$, $e_3(g_o)$ and $e_4(g_o)$. Then the points $X = (x,y)$ satisfying the inequalities

$$x + y \geq e_1(g_o)$$

$$x - y \geq e_2(g_o)$$

$$-x + y \geq e_3(g_o)$$

$$-x - y \geq e_4(g_o)$$

are solutions to problems being considered with $g_o > z_o$.

A lower bound for problem (2.2.1) assuming that $w_i = 0$ for some i is also given by this development. Let $B_{st}$ be the lower bound for the subproblem (2.8.2). Then a lower bound for problem (2.2.1) with some $w_i = 0$ is

$$B = \max \left[ B_{st}, \max_{\substack{1 \le i \le m \\ i \in W}} \left\{ g_i \right\} \right].$$

From Property 2.8.1 and the above discussion, B is a lower bound for problem (2.2.1).

# CHAPTER 3

# A MULTIFACILITY PROBLEM

## 3.1 Introduction

The problem considered in this chapter is to locate several new facilities with respect to a finite number of existing facilities so as to minimize the maximum cost, where the costs are proportional to the rectilinear distances between new and existing facilities or proportional to the rectilinear distances among new facilities. The problem also includes upper bound constraints on the rectilinear distances between new and existing facilities and among new facilities.

The transformation T, developed in Chapter 2, is applied to this problem and gives a decomposition into two independent one-dimensional minimax subproblems.

If there are costs or constraints among new facilities, each subproblem is formulated as an equivalent network weighted flow problem and is solved by determining a simple chain of maximum cost to weight ratio in the network. If there are no costs and no constraints among new facilities, each subproblem decomposes into one facility subproblems which are solved using the procedures of Chapter 2.

All optimal solutions to the original multifacility problem are obtained from the solutions to the subproblems.

## 3.2 Formulation

The multifacility minimax location problem is now formulated using the notation developed in Chapter 2.

Let $X_j$, $1 \leq j \leq n$, denote the n points in $E_2$ to be determined for the location of n new facilities. Let S be a finite set in $E_2$ of m points $P_i$, $1 \leq i \leq m$, representing the location of existing facilities.

For each point $P_i$ and $X_j$ let $d_{ji}$ be a positive constant which is an upper bound on the rectilinear distance between $P_i$ and $X_j$. Also let $c_{jk}$ be a positive constant which is an upper bound on the rectilinear distance between the pair of new facility locations $X_j$ and $X_k$ for $1 \leq j < k \leq n$.

The objective function is a function of the n variables $X_j$ in $E_2$, $1 \leq j \leq n$, and is the maximum of all expressions which are proportional to the rectilinear distance between each pair of new facility locations $X_j$ and $X_k$ and between each new facility location $X_j$ and each existing facility location $P_i$. Let $w_{ji}$ be a real nonnegative constant of proportionality for the rectilinear distance between each point $X_j$ and $P_i$ for $1 \leq j \leq n$, and $1 \leq i \leq m$. Let $v_{jk}$ be a real nonnegative constant of proportionality for the rectilinear distance between each pair of points $X_j$ and $X_k$ for $1 \leq j < k \leq n$.

Then the multifacility minimax location problem to be studied in this chapter can be stated as

$$\min_{X_1,\ldots,X_n \in E_2} \max \left[ \max_{1 \le j < k \le n} \left\{ v_{jk} \| X_j - X_k \|_1 \right\}, \max_{\substack{1 \le j \le n \\ 1 \le i \le m}} \left\{ w_{ji} \| X_j - P_i \|_1 \right\} \right]$$

$$\text{s.t.} \quad \| X_j - X_k \|_1 \le c_{jk} \qquad 1 \le j < k \le n$$

$$(3.2.1)$$

$$\| X_j - P_i \|_1 \le d_{ji} \qquad 1 \le j \le n, \quad 1 \le i \le m.$$

As before, the unconstrained multifacility problem, which will be referred to as problem (3.2.2), is a special case of problem (3.2.1) obtained by letting the upper bounds $c_{jk}$ and $d_{ji}$ be large.

Thus each of the n new facilities is to be located with respect to the m existing facilities and also with respect to the other new facilities. The location of the point $X_j$ may depend on the location of some point $X_k$ because of the terms involving $v_{jk}$ in the objective function or because of the constraint with upper bound $c_{jk}$. This dependence among new facility locations is made more explicit by the following definitions which were developed in [14] for an unconstrained total cost multifacility location problem and are extended here to problem (3.2.1).

New facility locations $X_j$ and $X_k$ are said to be linked if $v_{jk}$ is positive or if there exists a constraint for $X_j$ and $X_k$ with $c_{jk}$ finite, and are said to be not linked if $v_{jk}$ is zero and $c_{jk}$ is infinite, that is, there is no constraint for $X_j$ and $X_k$.

The index set, V(j), of positive $v_{jk}$ for each $1 \le j \le n - 1$ is defined by

$$V(j) = \{k : 1 \le j < k \le n,\ v_{jk} > 0\} .$$

The index set, $C(j)$, for which there exists finite $c_{jk}$ for $1 \leq j \leq n - 1$ is defined by

$$C(j) = \{k : 1 \leq j < k \leq n, \quad 0 < c_{jk} < \infty\} .$$

It is assumed that each new facility location $X_j$ is linked with at least one other new facility location. If this assumption is not made, then the location of $X_j$ could be determined independently of the other new facility locations by considering a separate problem, and problem (3.2.1) would not be formulated properly.

In addition, a new facility location $X_j$ and an existing facility location $P_i$ are said to be linked if $w_{ji}$ is positive or if there is a constraint for $X_j$ and $P_i$ with $d_{ji}$ finite, and are said to be not linked otherwise. The index set, $W(j)$, of positive $w_{ji}$ for each $1 \leq j \leq n$ is defined by

$$W(j) = \{i : 1 \leq i \leq m, \ w_{ji} > 0\} .$$

The index set, $D(j)$, for which there exist finite $d_{ji}$ for $1 \leq j \leq n$ is defined by

$$D(j) = \{i : 1 \leq i \leq m, \quad 0 < d_{ji} < \infty\} .$$

The new facility location $X_{j_1}$ is said to be chained if there exists a sequence of new facility locations $X_{j_1}, X_{j_2}, \ldots, X_{j_p}$, such that $X_{j_k}$ and $X_{j_{k+1}}$ in this sequence are linked for $1 \leq k \leq p - 1$, and the new facility at $X_{j_p}$ is linked with an existing facility at $P_i$ for some i. It is assumed that all new facility locations are chained, for otherwise the unchained locations could be placed at the same point in the plane.

The chaining assumptions are used in [14] to prove that there exists an optimal solution to the total cost multifacility problem. In an analogous fashion, the above assumptions imply that if there is a feasible solution to problem (3.2.1), there is an optimal solution.

With these definitions the multifacility minimax location problem (3.2.1) becomes

$$\min_{X_1,\ldots,X_n \in E_2} \max\left[\max_{\substack{1\leq j\leq n-1\\ k\in V(j)}} \left\{v_{jk}\|X_j - X_k\|_1\right\},\ \max_{\substack{1\leq j\leq n\\ i\in W(j)}} \left\{w_{ji}\|X_j - P_i\|_1\right\}\right] \tag{3.2.3}$$

$$\text{s.t.}\quad \|X_j - X_k\|_1 \leq c_{jk} \qquad 1\leq j\leq n-1,\ k\in C(j)$$

$$\|X_j - P_i\|_1 \leq d_{ji} \qquad 1\leq j\leq n,\quad i\in D(j).$$

In order to simplify the notation and presentation in the remainder of this chapter, the development is given for problem (3.2.1) with the assumption that the constants $v_{jk}$ are positive and the $c_{jk}$ are finite for all $1 \leq j < k \leq n$, and that the constants $w_{ji}$ are positive and the $d_{ji}$ are finite for all $1 \leq j \leq n$, $1 \leq i \leq m$. Differences between this development and a development for the more general problem (3.2.3) will be pointed out in the sequel.

Notice that for $n = 1$, the multifacility problem (3.2.1) reduces to a one facility problem.

A generalization of problem (3.2.1) may be formulated to include constants $g_{ji}$ corresponding to $X_j$ and $P_i$ for $1 \leq j \leq n$ and $1 \leq i \leq m$; and $h_{jk}$ corresponding to $X_j$ and $X_k$ for $1 \leq j < k \leq n$:

$$\min_{X_1,\ldots,X_n \in E_2} \max \left[ \max_{1\le j<k\le n} \left\{ v_{jk} \|X_j - X_k\|_1 + h_{jk} \right\}, \max_{\substack{1\le i\le m \\ 1\le j\le n}} \left\{ w_{ji} \|X_j - P_i\|_1 + g_{ji} \right\} \right]$$

$$\begin{aligned} \text{s.t.} \quad & \|X_j - X_k\|_1 \le c_{jk} \qquad 1 \le j < k \le n \\ & \|X_j - P_i\|_1 \le d_{ji} \qquad 1 \le i \le m, \quad 1 \le j \le n. \end{aligned} \tag{3.2.4}$$

Most of the development in this chapter applies to problem (3.2.4) with specific assumptions for the $g_{ji}$ and $h_{jk}$. A discussion of problem (3.2.4) in terms of the properties and procedures developed for problem (3.2.1) is included in a final section of this chapter.

## 3.3 Applications

Applications of multifacility location problems occur in the same contexts as those discussed in Chapters 1 and 2.

As a specific example, suppose that n new machines are to be located in a plant with respect to m existing machines located at points $P_i$, $1 \le i \le m$. Travel on a system of orthogonal aisles is to take place between the new machines and existing machines, and among the new machines. Let $w_{ji}$ be an estimate of the number of trips per month between new machine j and existing machine i. Let $v_{jk}$ be an estimate of the number of trips per month between new machines j and k. It is desired to locate the n new machines so that the maximum estimated travel distance per month is minimized. This problem may be formulated as problem (3.2.2).

In other applications the number $w_{ji}$ and $v_{jk}$ may find different interpretations as constants per unit of distance between facilities. For example, it may be desired to minimize the maximum cost of travel between new and existing facilities and among new facilities; then the $w_{ji}$ and $v_{jk}$ may be interpreted as costs per unit of distance. These numbers may also find interpretations, other than in the example, as weights or levels of interaction between facilities.

## 3.4 Related Work

First it is noted that the analogous total cost multifacility location problem has received attention previously for both Euclidean distances [14,30] and rectilinear distances [11,2]. Also, the Euclidean distance multifacility minimax problem is the subject of a paper by Elzinga and Hearn [6], and Love and Wesolowsky [24].

There appear to be only two reports extant concerning the multifacility problem explicitly using the rectilinear distance. The unconstrained problem, (3.2.2), was formulated independently by Francis [16] and Wesolowsky [30]. In [16] it is shown that the unconstrained problem may be written as an equivalent linear program. Wesolowsky [30] also makes that observation, but only for the unconstrained problem in one dimension. He then suggests a solution procedure using parametric linear programming.

## 3.5 The Transformation and Decomposition

The transformation T developed in Chapter 2 is now applied to problem (3.2.1) to obtain the rotated problem.

$$\min_{X'_1,\ldots,X'_n \in E_2} \max\left[\max_{1\leq j<k\leq n}\left\{v_{jk}\|X'_j - X'_k\|_\infty\right\}, \max_{\substack{1\leq j\leq n\\ 1\leq i\leq m}}\left\{w_{ji}\|X'_j - P'_i\|_\infty\right\}\right]$$

$$\begin{aligned} \text{s.t.} \quad & \|X'_j - X'_k\|_\infty \leq c'_{jk} \qquad 1 \leq j < k \leq n \\ & \|X'_j - P'_i\|_\infty \leq d'_{ji} \qquad 1 \leq j \leq n,\ 1 \leq i \leq m \end{aligned} \tag{3.5.1}$$

where $X'_j = T(X_j)$, $c'_{jk} = c_{jk}/\sqrt{2}$ and $d'_{ji} = d_{ji}/\sqrt{2}$.

The following property is an extension of Property 2.5.2 and shows that problems (3.2.1) and (3.5.1) are equivalent.

Property 3.5.1. The set $\{X_1^o, \ldots, X_n^o\}$ is an optimal solution to problem (3.2.1) if and only if $\{T(X_1^o), \ldots, T(X_n^o)\}$ is an optimal solution to problem (3.5.1). Furthermore, if $z_o$ and $z_o'$ are the optimal objective function values of problem (3.2.1) and (3.5.1), respectively, then $z_o = z_o' \sqrt{2}$ .

Analogous to the development of the one facility problem, two subproblems are defined from the rotated problem (3.5.1) as follows:

$$\begin{aligned} \min_{x_1', \ldots, x_n' \epsilon E_1} \max & \left[ \max_{1 \leq j < k \leq n} \left\{ v_{jk} \left| x_j' - k_k' \right| \right\}, \max_{\substack{1 \leq j \leq n \\ 1 \leq i \leq m}} \left\{ w_{ji} \left| x_i' - a_i' \right| \right\} \right] \\ \text{s.t.} \quad & \left| x_j' - x_k' \right| \leq c_{jk}' \qquad 1 \leq j < k \leq n \\ & \left| x_j' - a_i' \right| \leq d_{ji}' \qquad 1 \leq j \leq n, \quad 1 \leq i \leq m \end{aligned} \tag{3.5.2}$$

and

$$\begin{aligned} \min_{y_1', \ldots, y_n' \epsilon E_1} \max & \left[ \max_{1 \leq j < k \leq n} \left\{ v_{jk} \left| y_j' - y_k' \right| \right\}, \max_{\substack{1 \leq j \leq n \\ 1 \leq i \leq m}} \left\{ w_{ji} \left| y_j' - b_i' \right| \right\} \right] \\ \text{s.t.} \quad & \left| y_j' - y_k' \right| \leq c_{jk}' \qquad 1 \leq j < k \leq n \\ & \left| y_i' - b_i' \right| \leq d_{ji}' \qquad 1 \leq j \leq n, \quad 1 \leq i \leq m. \end{aligned} \tag{3.5.3}$$

Each subproblem is a one-dimensional multifacility minimax location problem. The following property is an extension of Property 2.5.3 and relates the optimal solution of the subproblems to the rotated problem (3.5.1).

Property 3.5.2. If $\{x_1^{o'},\ldots,x_n^{o'}\}$ is an optimal solution to subproblem (3.5.2) with optimal objective function value $z_1'$ and if $\{y_1^{o'},\ldots,y_n^{o'}\}$ is an optimal solution to subproblem (3.5.3) with optimal objective function value $z_2'$, then $\{(x_1^{o'},y_1^{o'}),\ldots,(x_n^{o'},y_n^{o'})\}$ is an optimal solution to problem (3.5.1) with optimal objective function value $z_o' = \max(z_1',z_2')$.

Thus Properties 3.5.1 and 3.5.2 show that the optimal solutions to the multifacility problem (3.2.1) are obtained from the independent solutions to the subproblems (3.5.2) and (3.5.3). In the next section, the subproblems are formulated as network problems and properties are developed.

## 3.6 A Network Formulation

In this section it is shown that each subproblem, (3.5.2) and (3.5.3), may be formulated as equivalent network weighted flow problems. Properties of the network problems are developed that form a basis for a solution procedure. The subproblems have the same structure so that subsequent properties are developed only for problem (3.5.2), with analogous properties existing for (3.5.3).

The constraints between new and existing facilities in subproblem (3.5.2) are simplified by using the development in Section 2.6 and defining

$$d_j' = \max_{1\leq i\leq m} (-d_{ji}' + a_i')$$

$$D_j' = \min_{1\leq i\leq m} (d_{ji}' + a_i')$$

for $1 \leq j \leq n$. Then the subproblem (3.5.2) is equivalent to the problem

$$\min_{x'_1,\ldots,x'_n \in E_1} \max\left[\max_{1\le j<k\le n}\left\{v_{jk}\left|x'_j - x'_k\right|\right\}, \max_{\substack{1\le j\le n\\ 1\le i\le m}}\left\{w_{ji}\left|x'_j - a'_i\right|\right\}\right]$$

$$\begin{array}{llll} \text{s.t.} & \left|x'_j - x'_k\right| \le c'_{jk} & 1 \le j < k \le n & \\ & d'_j \le x'_j \le D'_j & 1 \le j \le n. & (3.6.1) \end{array}$$

From the chaining assumptions made in Section 3.2, if problem (3.6.1) has a feasible solution, it has an optimal solution.

Based on the discussion in Section 2.4, problem (3.6.1) is equivalent to the following linear program:

$$\min z'$$

$$\begin{array}{lllll} \text{s.t.} & (1a) & x'_j - x'_k & \ge -c'_{jk} & \\ & & & & 1 \le j < k \le n \\ & (1b) & -x'_j + x'_k & \ge -c'_{jk} & \\ & (2a) & x'_j & \ge d'_j & \\ & & & & 1 \le j \le n \\ & (2b) & -x'_j & \ge -D'_j & \\ & (3a) & x'_j - x'_k + z'/v_{jk} & \ge 0 & \\ & & & & 1 \le j < k \le n \\ & (3b) & -x'_j + x'_k + z'/v_{jk} & \ge 0 & \\ & (4a) & x'_j + z'/w_{ji} & \ge a'_i & \\ & & & & 1 \le j \le n, \quad 1 \le i \le m \\ & (4b) & -x'_j + z'/w_{ji} & \ge -a'_i & \\ & & z', x'_j \text{ unrestricted} & & 1 \le j \le n. \qquad (3.6.2) \end{array}$$

A dual problem to (3.6.2) is written as problem (3.6.3). The dual variables in (3.6.3) are defined in Table 3.1.

$$\max \sum_{j=1}^{n} \left[ -\sum_{k>j}^{n} c'_{jk}F_{jk} + d'_{j}F_{jt} + \sum_{i=1}^{m} a'_{i}f^{i}_{jt} - \sum_{k>j}^{n} c'_{jk}F_{kj} - D'_{j}F_{sj} - \sum_{i=1}^{m} a'_{i}f^{i}_{sj} \right]$$

$$\text{s.t.} \quad (1j) \quad \sum_{\substack{k=1\\k\neq j}}^{n} F_{jk} + F_{jt} + \sum_{\substack{k=1\\k\neq j}}^{n} f_{jk} + \sum_{i=1}^{m} f^{i}_{jt} - \sum_{\substack{k=i\\k\neq j}}^{n} F_{kj} - F_{sj} - \sum_{\substack{k=1\\k\neq j}}^{n} f_{kj} - \sum_{i=1}^{m} f^{i}_{sj} = 0$$

$$1 \leq j \leq n$$

$$(2) \quad \sum_{j=1}^{n}\sum_{k>j}^{n} f_{jk}/v_{jk} + \sum_{j=1}^{n}\sum_{i=1}^{m} f^{i}_{jt}/w_{ji} + \sum_{j=1}^{n}\sum_{k>j}^{n} f_{kj}/v_{jk} + \sum_{j=1}^{n}\sum_{i=1}^{m} f^{i}_{sj}/w_{ji} = 1$$

(3) all variables nonnegative (3.6.3)

$$(4) \quad \sum_{j=1}^{n} F_{sj} + \sum_{j=1}^{n}\sum_{i=1}^{m} f^{i}_{sj} = v$$

$$(5) \quad -\sum_{j=1}^{n} F_{jt} - \sum_{j=1}^{n}\sum_{i=1}^{m} f^{i}_{jt} = -v.$$

The constraints (1j), for $1 \leq j \leq n$, (2) and (3) in problem (3.6.3) are the dual constraints obtained from problem (3.6.2). The additional constraints (4) and (5) are appended to the original constraints (1j), (2) and (3), and the new variable v is introduced without changing the problem. The additional constraints are obtained from the sum of the constraints (1j), for $1 \leq j \leq n$, so that they are redundant in problem (3.6.3).

TABLE 3.1

DUAL VARIABLES

| Variable Name | Range of Indices | Corresponding Constraint |
|---|---|---|
| $F_{jk}$ | $1 \leq j < k \leq n$ | (1a) |
| $F_{kj}$ | $1 \leq j < k \leq n$ | (1b) |
| $F_{jt}$ | $1 \leq j \leq n$ | (2a) |
| $F_{sj}$ | $1 \leq j \leq n$ | (2b) |
| $f_{jk}$ | $1 \leq j < k \leq n$ | (3a) |
| $f_{kj}$ | $1 \leq j < k \leq n$ | (3b) |
| $f^{i}_{jt}$ | $1 \leq j \leq n, \quad 1 \leq i \leq m$ | (4a) |
| $f^{i}_{sj}$ | $1 \leq j \leq n, \quad 1 \leq i \leq m$ | (4b) |

The coefficient matrix of constraints (1j), $1 \leq j \leq n$, (4) and (5) is a node-arc incidence matrix of a network, denoted by N, with $n + 2$ nodes. The n constraints (1j), for $1 \leq j \leq n$, correspond to a complete subnetwork of n nodes j, called intermediate nodes, such that between each pair of nodes j and k there are two pairs of directed arcs. For any two intermediate nodes, arcs in a pair have the same direction but have opposite direction to the other pair of arcs; the arcs in a pair have costs $-c'_{jk}$ and zero, respectively.

Node $n+1$ corresponds to constraint (4) and is denoted as the source, s. Node $n+2$ corresponds to constraint (5) and is denoted as the sink, t. The source and sink are connected to the other n intermediate nodes in the following manner. For each intermediate node j of the complete network there are $m+1$ directed arcs from the source s to node j. Each of the first m arcs has cost $-a'_i$, $1 \leq i \leq m$, and the $(m+1)^{st}$ arc has cost $-D'_j$. There are also $m+1$ directed arcs from the node j to the sink t. Each of the first m arcs has cost $a'_i$, $1 \leq i \leq m$, and the $(m+1)^{st}$ arc has cost $d'_j$.

The variables in problem (3.6.3) will be interpreted as flow variables on the corresponding arcs in the network N. The objective of (3.6.3) is to determine the flow of maximum total cost in the network N.

Constraint (2) of problem (3.6.3) implies that there are weights associated with some arcs in N and that the sum of the weighted flow in these arcs must equal one. The arcs that do not have weights associated with them are the arcs from node s to node j with cost $-D'_j$ and the arcs from node j to node t with cost $d'_j$ for $1 \leq j \leq n$. Also the arcs between pairs of intermediate nodes j and k with cost $-c'_{jk}$ for all $1 \leq j < k \leq n$, do not have weights. All such arcs will be called unweighted arcs. The remaining arcs will be called weighted arcs.

Then problem (3.6.3) is to determine the flow in N of maximum total cost subject to the restriction that the total of the weighted flow in the weighted arcs must equal one.

For $n = 3$ and $m = 2$, the network N is illustrated in Figure 3.1. The number corresponding to each arc is the arc cost. Broken lines represent unweighted arcs, and unbroken lines represent weighted arcs.

The unweighted arcs in the network N correspond to constraints in the linear program (3.6.2) that are determined from the constraints of problem (3.6.1). Therefore, for the problem (3.2.3), where not all constraints were assumed to exist, unweighted arcs in the network N would only exist for these constraints that are included in (3.2.3). The network N associated with the unconstrained problem (3.2.2) would have no unweighted arcs.

Similarly, weighted arcs in the network N correspond to constraints in the linear program (3.6.2) that are determined from the objective function of problem (3.6.1). For the problem (3.2.3), weighted arcs in the network N would exist only for those terms in the objective function with positive $v_{jk}$ or $w_{ji}$.

For the one facility problem, obtained by letting $n = 1$, the network N has only one intermediate node, but still has a source and sink and connecting arcs as in the discussion above.

## 3.7 Properties of the Network Problem

After establishing some definitions, properties of solutions to problem (3.6.3) are developed. Define a chain [8] from node $i_p$ to node $i_q$ in a network to be a set of directed arcs

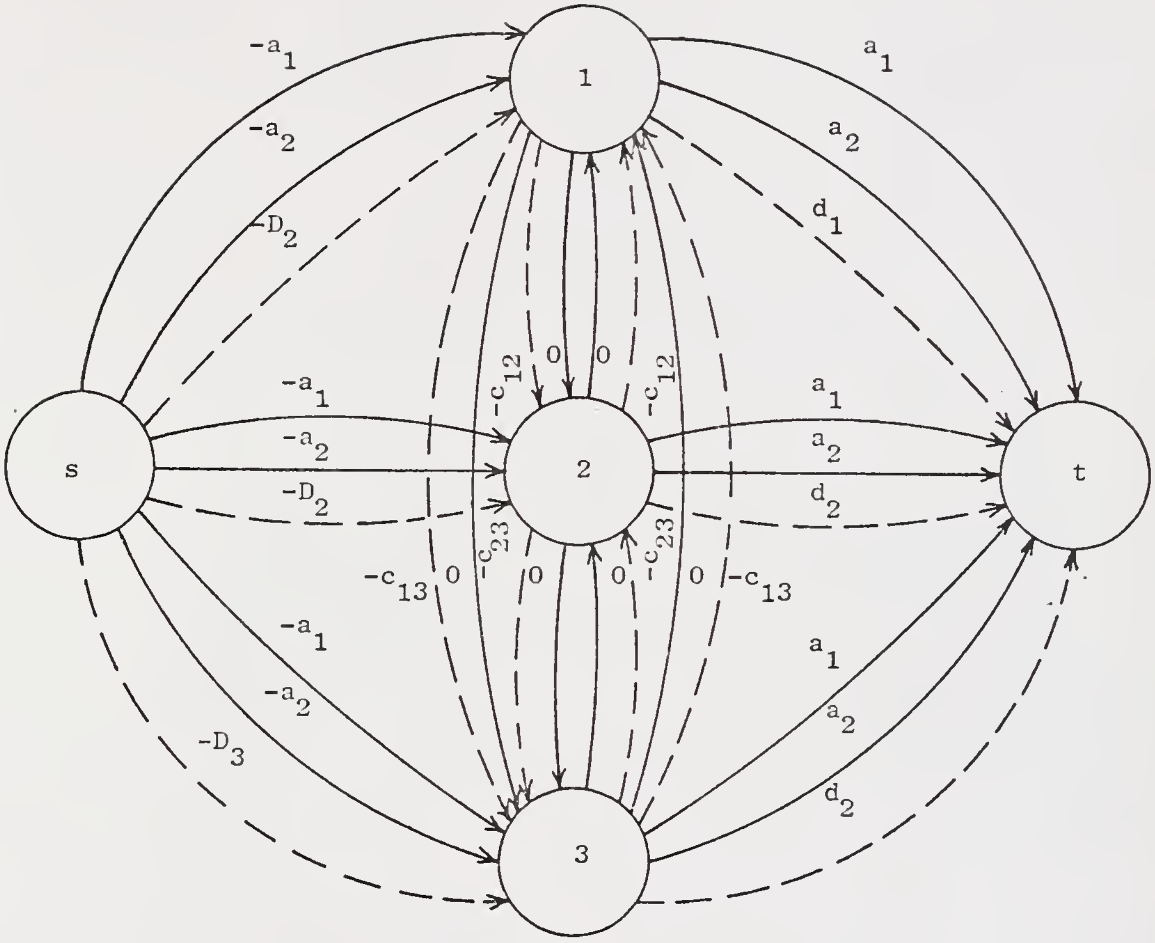

Figure 3.1 A Network N

$\{(i_p,i_1),(i_1,i_2),\ldots,(i_k,i_q)\}$ where $(i_j,i_{j+1})$ is a directed arc from node $i_j$ to node $i_{j+1}$ in the network. A cycle is a chain with $i_p = i_q$, and a simple chain is a chain that contains no cycles, that is, the $i_j$ are distinct. For any chain $C = \{(s,j_1),(j_1,j_2),\ldots,(j_p,t)\}$ in N from s to t, denote the total of the arc costs in C by T and denote the total of the arc weights in C by W. Also, let F represent the flow in each arc of the chain C.

Lemma 3.7.1. There exists a positive feasible flow in N with positive objective function value.

Proof: Of all the costs $-a'_i$, $1 \leq i \leq m$, for arcs between node s and node 1, let $-a'_1$ be the largest. Of all the costs $a'_i$ for arcs between 1 and node t, let $a'_m$ be the largest. Consider the chain $C = \{(s,1),(1,t)\}$ with arc costs $-a'_1$ and $a'_m$, respectively. Now $a'_m \neq a'_1$, for otherwise all the $a'_i$ would be equal and problem (3.2.1) would have the trivial solution $x'_j = a'_i$, for all $1 \leq j \leq n$. Therefore, $T = a'_m - a'_1 > 0$ and $W = w_{11} + w_{1m}$. A feasible solution is obtained by taking $f^1_{s1} = f^m_{1t} = \frac{1}{W} > 0$ and taking all other flow variables zero. This solution has objective function value $T/W > 0$.

Lemma 3.7.2. Any cycle in N has nonpositive total arc cost and consists only of arcs between intermediate nodes.

Proof: All coefficients in constraint (4) of problem (3.6.3) are positive so that there are no arcs directed into node s. Similarly, all coefficients in constraint (5) are negative so that there are no arcs directed out of the sink t. Hence any cycle must occur among intermediate nodes. From the objective function, all arc costs for arcs among intermediate nodes are nonpositive. Thus any cycle will have nonpositive total arc cost.

The next lemma characterizes a feasible flow in N for use in the proof of subsequent properties.

Lemma 3.7.3. Any feasible flow in N consists of either a cycle of flow, a chain of flow from s to t of unweighted arcs, a chain of flow from s to t of at least one weighted arc, or some combination of these.

If there is an unweighted chain in N of zero total arc cost, then any feasible solution to problem (3.6.3) can include any amount of flow on this chain without affecting the objective function value. To exclude from further consideration those feasible solutions that contain flow on such chains allows a more explicit presentation and causes no loss of generality. Therefore, it is assumed for the remainder of the development that any feasible solution has zero flow on unweighted chains of zero total arc cost.

Property 3.7.1. Problem (3.6.1) has no feasible solution if and only if there exists a chain of unweighted arcs in N which has a positive total arc cost.

Proof: Suppose there exists a chain $C_k$ in N of unweighted arcs with total arc cost $T_k > 0$. By Lemma 3.7.1 there exists a feasible flow on some chain $C_i$ with $T_i > 0$ and $W_i > 0$. Then putting a flow of $1/W_i$ on each arc of $C_i$ and putting infinite positive flow on $C_k$ is a feasible solution to problem (3.6.3) with unbounded objective function value. Since problem (3.6.2) is the linear programming dual of problem (3.6.3), problem (3.6.2) has no feasible solution.

A contrapositive proof is given for the converse statement. Suppose all chains from s to t of unweighted arcs in N have nonpositive total costs. Since problem (3.6.3) is a maximization problem, Lemma 3.7.1 and the above assumption imply that any optimal solution will have zero flow on all such chains. There will also be zero flow on all cycles, since cycles have nonpositive total arc cost by Lemma 3.7.2. By Lemma 3.7.3 all remaining feasible solutions will correspond to chains from s to t with at least one weighted arc in each chain. Then constraint (2) implies that flow on these chains is bounded. Thus the existence of a bounded feasible solution to problem (3.6.3) implies that there is an optimal solution to problem (3.6.3). Hence there exists a feasible solution to problem (3.6.2).

Property 3.7.2. If a feasible solution to problem (3.6.3) is optimal (with bounded objective function value), then the optimal solution consists of positive flow on simple chains from s to t with positive total arc weight.

Proof: An optimal solution to problem (3.6.3) with bounded objective function value implies that all chains of unweighted arcs in N have nonpositive total arc cost. If such a chain has zero total arc cost, by assumption there is zero flow on that chain. If such a chain has negative total arc cost, a positive flow on the chain will not be optimal. Thus, flow on unweighted chains of nonpositive total arc cost is zero in the optimal solution.

Also, all cycles have nonpositive total cost by Lemma 3.7.2 and will have zero flow in an optimal solution.

By Lemma 3.7.1 an optimal solution will have positive flow on some arcs. Thus by Lemma 3.7.3 an optimal solution must correspond to positive flow on chains from s to t, each of which contains at least one weighted arc. These chains do not contain cycles since all cycles have zero flow. Thus an optimal solution consists of positive flow on simple chains from s to t that contain at least one weighted arc.

Property 3.7.3. Suppose problem (3.6.1) has a feasible solution and let $C_1, C_2, \ldots, C_r$ be all the simple chains from s to t in N with chain $C_i$, having total arc cost $T_i$ and total arc weight $W_i > 0$. Let chain $C_k$ have the property that

$$T_k/W_k \geq T_i/W_i$$

for $1 \leq i \leq r$. Then putting a flow of value $F_k = 1/W_k$ on all arcs in $C_k$ and zero flow elsewhere is an optimal solution to problem (3.6.3).

Proof: Since problem (3.6.1) has a feasible solution, problem (3.6.3) has a bounded optimal solution and ay Property 3.7.2 any optimal solution will be determined by simple chains with positive total arc weights.

Suppose that positive flow on the simple chains $C_{j_1}, \ldots, C_{j_p}$ is an optimal solution to problem (3.6.3) with objective function value and total weighted flow given by

$$\sum_{i=1}^{p} F_{j_i} T_{j_i} \quad \text{and} \quad \sum_{i=1}^{p} F_{j_i} W_{j_i} = 1,$$

respectively. But

$$\sum_{i=1}^{p} F_{j_i} T_{j_i} = \sum_{i=1}^{p} F_{j_i} W_{j_i} \frac{T_{j_i}}{W_{j_i}}$$

$$\leq \frac{T_k}{W_k} \sum_{i=1}^{p} F_{j_i} W_{j_i}$$

$$= \frac{T_k}{W_k} = F_k T_k \ .$$

Thus the flow $F_k$ on chain $C_k$ is also an optimal solution to problem (3.6.3).

Property 3.7.3 provides a basis for a solution procedure to problem (3.6.3). That is, if problem (3.6.3) has a bounded objective function value, an optimal solution is determined by the simple chain, $C_k$, from s to t with $T_k > 0$ and maximum ratio $T_k/W_k$, referred to as the cost to weight ratio. A procedure for determining a chain in N of maximum cost to weight ratio is developed next. This procedure also determines if there is an unbounded solution to problem (3.6.3), that is, if there is a chain of unweighted arcs with positive total arc cost.

Similar optimization problems concerning cycles in general networks have received previous attention in the literature and may be considered as special cases of hyperbolic programming problems.

Dantzig, Blattner and Rao [5] consider a ship routing problem that is formulated as a problem of determining a cycle in a network of minimum cost to time ratio. Their solution procedure is a variation of the simplex algorithm. Fox [9] considers the same problem and solves it using a parametric application of the out-of-kilter algorithm.

A problem equivalent to problem (3.6.3) may be formulated and interpreted as finding a cycle in a corresponding network of maximum cost to weight ratio from those cycles with positive total weight. Using such a formulation, some developments in [5] and [9] would apply. However, an exposition in terms of chains is chosen because of the presence of the unweighted arcs and the special structure of the network N.

Notice that for the unconstrained problem (3.6.2), there are no unweighted arcs in the network N and the dual problem is always bounded.

Consider a network N' with the same node-arc structure as N but with different arc costs. The arc costs in N' are obtained from the arc costs of N by the following procedure. Subtract $\lambda/w_{ji}$ from the corresponding arc costs $a_i'$ and $-a_i'$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Subtract $\lambda/v_{jk}$ from the zero arc cost of corresponding intermediate arcs. Then multiply all arc costs by -1. The arc costs for arcs in N' are defined as follows:

$$c_{pq} + \lambda W_{pq} = \begin{cases} a'_i + \lambda/w_{ji} & p = s,\ q = j,\ 1 \le i \le m \\ D'_j & p = s,\ q = j,\ i = m+1 \end{cases} \Bigg\} \ 1 \le j \le n$$

$$\begin{cases} -a'_i + \lambda/w_{ji} & p = j,\ q = t,\ 1 \le i \le m \\ -d'_j & p = j,\ q = t,\ i = m+1 \end{cases} \Bigg\} \ 1 \le j \le n$$

$$\begin{cases} \lambda/v_{jk} & p = j,\ q = k, \\ c_{jk} & p = j,\ q = k, \end{cases} \Bigg\} \ \begin{array}{l} 1 \le j \le n \\ 1 \le k \le n \\ j \ne k \ . \end{array}$$

The network N' corresponding to the network N in Figure 3.1 is given in Figure 3.2.

Every simple chain from s to t in N corresponds to a unique simple chain from s to t in N' having the same node-arc progression but different arc costs. Denote a simple chain in N by $C_i$ and the corresponding simple chain in N' by $C'_i$. Any chain $C'_i$ in N' has total arc cost denoted by $T'_i$ and defined by

$$T'_i = \sum_{(p,q)\in C'_i} (c_{pq} + \lambda W_{pq}) \ .$$

If the corresponding chain $C_i$ has positive total arc weight, the total arc cost of $C'_i$ is a strictly increasing function of $\lambda$. For each chain $C'_i$ whose corresponding chain $C_i$ has positive total arc weight, define the root value of $C'_i$, denoted $\lambda_i$, as the value of $\lambda$ for which chain $C'_i$ has zero total arc cost. That is,

$$\sum_{(p,q)\in C'_i} (c_{pq} + \lambda_i W_{pq}) = 0 \ .$$

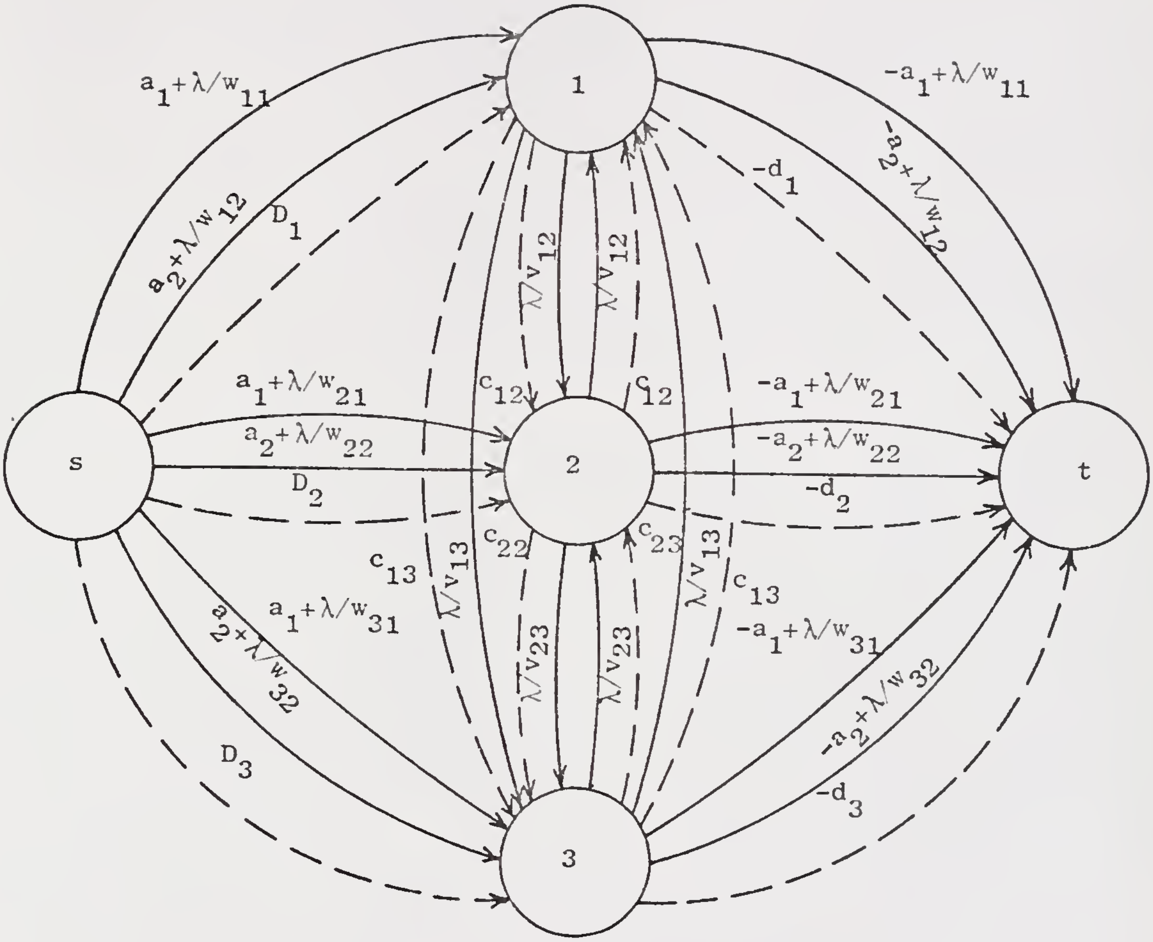

Figure 3.2 A Network N'

Solving for $\lambda_i$ gives

$$\lambda_i = \sum_{(p,q)\in C_i'} - c_{pq} \Big/ \sum_{(p,q)\in C_i'} W_{pq} = \frac{T_i}{W_i}$$

by the definition of $c_{pq}$ and $W_{pq}$ in terms of the chain $C_i$ in N.

Therefore, finding the chain in N with positive total arc weight and maximum cost to weight ratio is equivalent to finding the corresponding chain in N' having the largest root value. Denote the largest root value by $\lambda^*$.

Property 3.7.4. If problem (3.6.3) has an optimal solution with bounded objective function value, say $z_1'$, then $\lambda^* = z_1'$.

Proof: Let chain $C_o'$ correspond to $\lambda^*$, that is,

$$\lambda^* = \sum_{(p,q)\in C_o'} -c_{pq} \Big/ \sum_{(p,q)\in C_o'} W_{pq} .$$

The corresponding chain $C_o$ in N has total arc weight $W_o$ and total arc cost $T_o$ with

$$\lambda^* = \frac{T_o}{W_o} \geq \frac{T_i}{W_i}$$

for all chains $C_i$ in N. By Property 3.7.3, the optimal flow on $C_o$ is $F_o = 1/W_o$. Then the optimal objective function value is $F_oT_o = T_o/W_o = \lambda^*$.

Consideration is now given to chains in N of unweighted arcs. If a chain $C'_i$ in N' corresponds to a chain $C_i$ in N of unweighted arcs, then $C'_i$ has total arc cost

$$\sum_{(p,q)\epsilon C'_i} c_{pq}$$

which is a constant with respect to $\lambda$, since all $W_{pq}$ are zero in $C'_i$. If the constant total arc cost of $C'_i$ is negative, then $C_i$ in N is a chain of unweighted arcs with positive total arc cost. By Property 3.7.1, problem (3.6.1) has no feasible solution.

## 3.8 A Solution Procedure

Using the properties of the previous section, a solution procedure is developed for problem (3.6.3). The following algorithm determines a chain C' in N', having the largest root value of all chains in N', or it determines if there is a chain in N' of constant total arc cost which is negative. Therefore the algorithm either determines the chain in N corresponding to the optimal solution, or it determines that there is no bounded optimal solution.

### Algorithm

1. Set $i = 0$, $\lambda_i = 0$.

2. Using the current value of $\lambda_i$, find the chain from s to t in N' of minimum total arc cost. This may be done by using any shortest chain algorithm that handles both positive and negative arc costs, for example the algorithm given in Ford and Fulkerson [8]. Use the arc costs determined by the current value of $\lambda_i$ as the arc lengths in

the shortest chain algorithm. Denote the chain of minimum total arc cost by $C'_{i+1}$ and its total arc costs by $T'_{i+1}$. If $T'_{i+1} = 0$, stop; $C'_i$ is optimal and $\lambda_i$ is the largest root value. If $T'_i < 0$ and $\sum_{(p,q)\epsilon C'_{i+1}} W_{pq} > 0$, go to step 3. If $T'_i < 0$ and $\sum_{(p,q)\epsilon C'_{i+1}} W_{pq} = 0$, stop; problem (3.6.1) is infeasible.

3. Set $T'_{i+1} = 0$, compute $\lambda_{i+1}$ by

$$\lambda_{i+1} = \sum_{(p,q)\epsilon C'_{i+1}} - c_{pq} \Big/ \sum_{(p,q)\epsilon C'_{i+1}} W_{pq} .$$

Set i + 1 = i, go to step 2.

The following remarks show that the algorithm converges finitely to the desired chain.

Remark 3.8.1. $\lambda_{i+1} > \lambda_i$.

Proof: If $i = 0$, then $\lambda_o = 0$, and by Lemma 3.7.1 there exists some chain in N with positive total arc cost and positive total arc weight. Then the corresponding chain in N' has negative total cost. Therefore, with $\lambda_o = 0$, the minimum arc cost chain, denoted by $C'_1$, has $T'_1 < 0$.

Next, suppose that $i \geq 0$, and, using the current value of $\lambda_i$, chain $C'_{i+1}$ has $T'_{i+1} < 0$ and $\sum_{(p,q)\epsilon C'_{i+1}} W_{pq} > 0$. Then by the definition of $T'_{i+1}$ and $\lambda_{i+1}$

$$\lambda_i < \sum_{(p,q)\epsilon C'_{i+1}} - c_{pq} \Big/ \sum_{(p,q)\epsilon C'_{i+1}} W_{pq} = \lambda_{i+1}$$

and the assertion holds. If $T'_{i+1} < 0$ and $\sum_{(p,q)\epsilon C'_{i+1}} W_{pq} = 0$, then by step 2 the algorithm stops without specifying $\lambda_{i+1}$. If $T'_{i+1} = 0$, then the algorithm stops at optimality without specifying $\lambda_{i+1}$.

Remark 3.8.2. The algorithm is finite.

Proof: At each iteration the algorithm either stops or a new root value $\lambda_{i+1} > \lambda_i$ is determined. Since the number of chains is finite, and any chain is considered at most once, the largest $\lambda_i$ will be determined in a finite number of iterations.

Remark 3.8.3. The chain identified at the termination of the algorithm corresponds to a chain in N that is either an unweighted chain of positive total arc cost or a weighted chain of maximum cost to weight ratio.

Proof: Suppose the algorithm stops at step 2 by determining a chain $C'_{i+1}$ with $T'_{i+1} = 0$. Then $i \geq 1$ and the previous chain $C'_i$ has, for the current root value $\lambda_i$, total arc cost $T'_i = 0$ and $\sum_{(p,q)\epsilon C'_i} W_{pq} > 0$. Since $T'_i$ is the minimum arc cost, $T'_i \leq T'_j$ for all other chains $C'_j$ in N', using the value $\lambda_i$. Therefore, in the network N, $T_i/W_i \geq T_i/W_j$ for all chains $C_j$ in N, and $C_i$ is a weighted chain of maximum cost to weight ratio.

Suppose the algorithm stops at step 2 because the chain $C'_{i+1}$ has $T'_{i+1} < 0$ but $\sum_{(p,q)\epsilon C'_{i+1}} W_{pq} = 0$. Then the chain $C_{i+1}$ in N is an unweighted chain of positive total arc cost.

If problem (3.6.3) has an optimal solution with bounded objective function value, then the algorithm identifies a chain $C_i$ in N that determines an optimal solution to (3.6.3). However, it is desired to obtain all optimal solutions to the primal problem (3.6.1). If a labeling algorithm is used in step 2 to find the minimum cost chain, one optimal solution to problem (3.6.1) can be determined from the node labels (dual variables). The node labels also indicate a procedure to determine alternative solutions.

The dual of the minimum cost chain problem for the network N', at iteration i, is written below with the following change of variables; each variable is multiplied by -1.

$$
\begin{array}{lllll}
\min\ s - t & & & & \\
\text{s.t.} & x'_j - x'_k & \geq & -c'_{jk} & \\
 & -x'_j + x'_k & \geq & -c'_{jk} & 1 \leq j < k \leq n \\
 & & & & \\
 & x'_j \qquad - t & \geq & d'_j & \\
 & -x'_j \quad + s & \geq & -D'_j & 1 \leq j \leq n \\
 & & & & \\
 & x'_j - x'_k & \geq & -\lambda_i / v_{jk} & \\
 & -x'_j + x'_k & \geq & -\lambda_i / v_{jk} & 1 \leq j < k \leq n \\
 & & & & \\
 & x'_j \qquad - t & \geq & a'_i - \lambda_i / w_{ji} & \\
 & -x'_j \quad + s & \geq & -a'_i - \lambda_i / w_{ji} & 1 \leq j \leq n\,.
\end{array} \tag{3.8.1}
$$

At the last iteration, $\lambda_i = \lambda^*$ and by Property 3.7.4, $\lambda^* = z'_i$, the optimal objective function of problem (3.6.1). Since the source s has no arcs directed into it, the minimum cost chain algorithm gives $s = 0$. Since the minimum cost chain at the terminal iteration has zero cost, $t = 0$. This implies that, at optimality, the constraints to (3.8.1) are equivalent to the constraints of problem (3.6.2). Therefore the node labels in appropriate shortest chain algorithms determine the primal variables $x'_j$, $1 \leq j \leq n$, that form an optimal solution to problem (3.6.1).

If the minimum cost chain does not pass through some node, say j, then flow into and out of node j is zero. This corresponds to a degenerate solution to problem (3.6.3) so that its dual, problem (3.6.2), has alternative solutions. The alternative solution values for $x'_j$ may be determined using those constraints in (3.8.1) that involve $x'_j$ and using the value $z'_1 = \lambda^*$. Then from the development in Section 2.7, the alternative values for $x'_j$ must satisfy

$$\begin{aligned} x'_j &\geq \max \left[ \max_{1 \leq j \leq n} d'_j \, , \max_{\substack{1 \leq j \leq n \\ 1 \leq i \leq m}} \{a'_i - z'_1/w_{ji}\} \right] \\ x'_j &\leq \min \left[ \min_{1 \leq j \leq n} D'_j \, , \min_{\substack{1 \leq j \leq n \\ 1 \leq j \leq m}} \{a'_i + z'_1/w_{ji}\} \right] \end{aligned} \qquad (3.8.2)$$

and

$$|x'_j - x'_k| \leq \min \{c'_{jk}, z'_1/v_{jk}\} \quad 1 \leq k \leq n, \; j \neq k. \qquad (3.8.3)$$

The expressions (3.8.2) may be used to determine an interval for $x'_j$. However, expression (3.8.3) is an additional constraint on alternative solutions that determines subintervals depending on other variables $x'_k$.

Using the optimal solutions to the subproblems (3.5.2) and (3.5.3), a procedure is given based on Section 2.7 for finding alternative solutions to the original multifacility problem (3.5.1).

Solutions $\{x_1^{o'},\ldots,x_n^{o'}\}$ and $\{y_1^{o'},\ldots,y_n^{o'}\}$ to subproblems (3.5.2) and (3.5.3), respectively, are obtained using the algorithm. Let $z_1'$ and $z_2'$ be the respective optimal objective function value of the subproblems. Then Property 3.5.2 gives $z_o' = \max(z_1', z_2')$ as the optimal objective function value of (3.5.1), and gives a procedure for determining one optimal solution to (3.5.1). As in Section 2.7, there are three cases that determine all alternative solutions.

Case 1. $z_o' = z_1' = z_2'$. Then all the solutions $\{x_1',\ldots,x_n'\}$ to problem (3.5.2) paired with all the solutions $\{y_1',\ldots,y_n'\}$ to problem (3.5.3) give all the optimal solutions $\{(x_1',y_1'),\ldots,(x_n',y_n')\}$ to problem (3.5.1).

Case 2. $z_o' = z_1' > z_2'$. Then for all the points $y_j'$ for $1 \le j \le n$, that satisfy the expressions

$$\begin{gathered}
y_j' \ge \max\left[\max_{1\le j\le n} d_j' \,,\ \max_{\substack{1\le j\le n\\ 1\le i\le m}} \{b_i' - z_o'/w_{ji}\}\right] \\
y_j' \le \min\left[\min_{1\le j\le n} D_j' \,,\ \min_{\substack{1\le j\le n\\ 1\le i\le m}} \{b_i' + z_o'/w_{ji}\}\right]
\end{gathered} \tag{3.8.4}$$

$$|y_j' - y_k'| \le \min\{c_{jk}', z_o'/v_{jk}\} \quad 1 \le k \le n, \quad j \ne k \tag{3.8.5}$$

the points $(x_j^{o'}, y_j')$, $1 \le j \le n$, are alternative solutions to (3.5.1), for all alternative solutions $\{x_1^{o'},\ldots,x_n^{o'}\}$ to problem (3.5.3).

Case 3. $z'_o = z'_2 > z'_1$. Then for all the points $x'_j$, $1 \leq j \leq n$, that satisfy expressions (3.8.2) and (3.8.3), with $z'_o$ replacing $z'_1$; the points $(x'_j, y^{o'}_j)$, $1 \leq j \leq n$, are alternative solutions to (3.5.1), for all alternative solutions $\{y^{o'}_1, \ldots, y^{o'}_n\}$ to problem (3.5.2).

By Property 3.5.1, optimal solutions to the problem (3.2.1) are obtained by taking the inverse rotation, $T^{-1}$, of the solutions to (3.5.1).

## 3.9 A Special Case

A special case of problem (3.2.3) is considered in this section. It is obtained by assuming that none of the new facilities are linked; that is, all $v_{jk}$ are zero and there are no constraints for the new facilities. The problem is

$$\min_{X_1, \ldots, X_n \varepsilon E_2} \max_{\substack{1 \leq j \leq n \\ i \varepsilon W(j)}} \left\{ w_{ji} \| X_j - P_i \|_1 \right\}$$

$$\text{s.t.} \quad \| X_j - P_i \|_1 \leq d_{ji} \quad 1 \leq j \leq n, \ i \ \varepsilon \ D(j). \tag{3.9.1}$$

It is assumed that each new facility $X_j$ is linked to some existing facility $P_i$; otherwise, $X_j$ could be located independently of $P_i$. It is also assumed that each existing facility $P_i$ is linked to some new facility $X_j$; otherwise, $P_i$ could be deleted from the problem. These assumptions exclude from consideration problems that are not well formulated.

Then problem (3.9.1) may be treated as n independent one facility subproblems:

$$\min_{X_j \in E_2} \max_{i \in W(j)} \left\{ w_{ji} \| X_j - P_i \|_1 \right\}$$

$$\text{s.t.} \quad \| X_j - P_i \|_1 \le d_{ji} \qquad i \in D(j) \tag{3.9.2}$$

for $1 \le j \le n$. Each of these problems is solved using the development of Chapter 2, and from these solutions, all alternative solutions are obtained.

The transformation T is applied to each of the problems (3.9.2) and its subproblems are defined for $1 \le j \le n$. Let $z'_{1j}$ and $z'_{2j}$ be the optimal objective function values for the $j^{th}$ subproblems in $x'_j$ and $y'_j$, respectively. Then $z_o = \sqrt{2} \max_{1 \le j \le n} \{z'_{1j}, z'_{2j}\}$ is the optimal objective function value of problem (3.9.1). All alternative solutions can be determined from the following procedure.

Points $X_j$ that are alternative solutions to the $j^{th}$ subproblem (3.9.2) are those points satisfying

$$w_{ji} \| X_j - P_i \|_1 \le z_o \qquad i \in W(j)$$

$$\| X_j - P_i \|_1 \le d_{ji} \qquad i \in D(j) .$$

Based on the development in Section 2.8, define

$$e_1(z_o, j) = \max \left[ \max_{i \in W(j)} (a_i + b_i - z_o/w_{ji}), \max_{i \in D(j)} (a_i + b_i - d_{ji}) \right]$$

$$e_2(z_o, j) = \max \left[ \max_{i \in W(j)} (a_i - b_i - z_o/w_{ji}), \max_{i \in D(j)} (a_i - b_i - d_{ji}) \right]$$

$$e_3(z_o,j) = \max\left[\max_{i\in W(j)} (-a_i + b_i - z_o/w_{ji}),\ \max_{i\in D(j)} (-a_i + b_i - d_{ji})\right]$$

$$e_4(z_o,j) = \max\left[\max_{i\in W(j)} (-a_i - b_i - z_o/w_{ji}),\ \max_{i\in D(j)} (-a_i - b_i - d_{ji})\right].$$

Then the four inequalities

$$x_j + y_j \geq e_1(z_o, j)$$

$$x_j - y_j \geq e_2(z_o, j)$$

$$-x_j + y_j \geq e_3(z_o, j)$$

$$-x_j - y_j \geq e_4(z_o, j)$$

for $1 \leq j \leq n$, determine all alternative optimal solutions to problem (3.9.1).

### 3.10 An Extended Problem

This section contains a discussion of the extended problem, (3.2.4). The development of Sections 3.5 and 3.6 apply to problem (3.2.4) without any additional assumptions on the $g_{ji}$ and $h_{jk}$. That is, two subproblems, corresponding to (3.5.2) and (3.2.3), may be defined and each subproblem has a dual with a corresponding network. Let $N''$ denote the network of the subproblem in $x'_1,\ldots,x'_n$. The network $N''$ has the same node-arc structure as N, but with different arc cost. Arcs in N incident to node j with costs $-a'_i$ and $a'_i$ have corresponding arc costs $-a'_i + g'_{ji}/w_{ji}$ and $a'_i + g'_{ji}/w_i$, respectively, in $N''$ for $1 \leq j \leq n$, $1 \leq i \leq m$. An arc with a cost of zero in N has a corresponding arc cost $h_{jk}/v_{jk}$ in $N''$ for $1 \leq j < k \leq n$. Unweighted arcs in $N''$ have the same arc costs as corresponding unweighted arcs in N.

Assumptions must be made on the $g_{ji}$ and $h_{jk}$ in order for the properties and lemmas of Section 3.7 to hold for the network $N'$ and its corresponding subproblem.

If there are positive $h_{jk}$, then the weighted arcs between intermediate nodes will have positive cost. Therefore the network used in the algorithm may have cycles of negative total cost and the shortest chain algorithm cannot be used. Hence the above procedure will not apply if there exist positive $h_{jk}$ such that a cycle in $N'$ has positive total arc cost. If negative constants $h_{jk}$ exist such that a cycle has negative total cost, there will always be zero flow on that cycle at optimality. The corresponding arcs between intermediate nodes will not be in the optimal solution.

If the $g_{ji}$ are nonpositive, then Lemma 3.7.1 may not hold. Nevertheless, the results in [5] appear to extend so that the optimal solution to the network problem still corresponds to simple chains in $N'$. Additional research is required to completely extend the procedure to nonpositive $g_{ji}$. If assumptions are made on the constants $h_{jk}$ and $g_{jk}$ to avoid both of the above situations, then the properties and solution procedures developed for problem (3.2.1) also hold for problem (3.2.4).

# CHAPTER 4

# A p-CENTER LOCATION PROBLEM

## 4.1 Introduction

Considered in this chapter is a p-center problem, which is to locate p new facilities in the plane, for a given value of p, with respect to a finite number of existing facilities. There is a cost defined for each existing facility which is the minimum, over the new facilities, of linear functions of the rectilinear distance between the existing facilities and each new facility. The problem is to locate the new facilities so that the maximum cost, over the existing facilities, is minimized.

An example of a p-center problem is to locate p points in the plane so that the maximum rectilinear distance between each existing point and the nearest new point is minimized.

Candidates for the optimal objective function values of the p-center problems for all values of p are determined. For each candidate, a set covering problem is formulated whose solution determines a feasible solution to a p-center problem for some value of p. By solving a set covering problem for each candidate, optimal solutions to the p-center problems for all values of p are determined.

## 4.2 Formulation

Suppose p is a fixed integer. Let $X_j$, $1 \leq j \leq p$, denote the points in $E_2$ to be determined for the location of p new facilities called centers and let the set $\underline{X}_p = \{X_1,\ldots,X_p\}$ denote a p-center. As in the previous chapters, S will be a finite set in $E_2$ of m points $P_i$, $1 \leq i \leq m$, representing the locations of existing facilities. For each point $P_i$, let $w_i$ be any real positive constant and let $g_i$ be any real constant.

For each i and j, $1 \leq i \leq m$, $1 \leq j \leq p$, define a continuous function in the p variables $X_j$ by

$$f_{ji}(X_1,\ldots,X_p) = w_i \| X_j - P_i \|_1 + g_i \ .$$

The p-center problem, to be studied in this chapter, may be written as follows:

$$\min_{X_1,\ldots,X_p} \ \max_{1 \leq i \leq m} \ \min_{1 \leq j \leq p} \{f_{ji}(X_1,\ldots,X_p)\}. \tag{4.2.1}$$

To show that an optimal solution to (4.2.1) exists, note that the function

$$f(X_1,\ldots,X_p) = \max_{1 \leq i \leq m} \ \min_{1 \leq j \leq p} f_{ji}(X_1,\ldots,X_p)$$

is continuous. Now consider the region in $E_2$ determined by the rectangle containing the points $P_i$ with sides of minimum length and parallel to the x and y axes, respectively. Any point $X_j$ outside this region may be moved, with no increase in the objective function value, in directions parallel to the x or y axes until $X_j$ is contained in the rectangle.

Therefore it is sufficient to show that there is an optimal solution within the rectangle. Since the rectangle is compact in $E_2$, the set in $E_{2p}$ formed by p cross-products of these rectangles is also compact. Therefore $f(X_1,\dots,X_p)$ attains its minimum value in this compact set in $E_{2p}$.

Notice that for p = 1, problem (4.2.1) is a special case of a one facility minimax location problem considered in Chapter 2.

The p-center problem may be written equivalently as the constrained problem:

$$\begin{aligned} &\min z \\ &\text{s.t.} \quad \min_{1\le j\le p} w_i \| X_j - P_i \| + g_i \le z \qquad 1 \le i \le m. \end{aligned} \tag{4.2.2}$$

A geometrical motivation for the p-center problem may be given using the following definition. A diamond centered at $P_i$ and of value $z^*$ denoted by $D(P_i,z^*)$ and is defined by

$$D(P_i,z^*) = \{X \in E_2 : w_i \| X - P_i \|_1 + g_i \le z^*\} .$$

Then, in reference to a diamond as defined in [15], $D(P_i,z^*)$ has radius $(z^* - g_i)/w_i$.

The p-center problem is to determine the smallest value z for the diamonds $D(P_i,z)$ and to locate p points $X_j$ within these diamonds, so that for each i, $1 \le i \le m$, $X_j \in D(P_i,z)$ for some j.

Suppose that a value $z^*$ is given. Then a problem related to the p-center problem is to determine a set $\underline{X}$ composed of a minimum number of centers that satisfy the constraints of the p-center problem

(4.2.2) with respect to $z^*$, that is, determine $\underline{X}$ such that for each i, $1 \leq i \leq m$ and for some $X_j \in \underline{X}$,

$$X_j \in D(P_i, z^*).$$

This problem is now formulated as a set covering problem and its relation to problem (4.2.1) is established in Property 4.2.1.

Consider the intersections of the diamonds $D(P_i, z^*)$ that are determined by all combinations of the points $P_i$ taken one at a time, two at a time, and so forth, up to m at a time. Let $M(z^*)$ be the number of different intersections that are nonempty. For each j, $1 \leq j \leq M(z^*)$, denote the $j^{th}$ nonempty intersection by $A_j(z^*)$. Let $I_j(z^*)$ be the index set of i such that the intersection of diamonds $D(P_i, z^*)$ equals $A_j(z^*)$. Then

$$A_j(z^*) = \bigcap_{i \in I_j(z^*)} D(P_i, z^*) .$$

For any $X_j$ in the intersection $A_j(z^*)$, $X_j$ satisfies the $i^{th}$ constraint for $i \in I_j(z^*)$ of problem (4.2.2). The problem of interest is to determine the minimum number of intersections $A_j(z^*)$ so that locating a center $X_j$ in $A_j(z^*)$ has the property that each diamond contains a center.

Define a variable $x_j$ for each intersection $A_j(z^*)$ by

$$x_j = \begin{cases} 1 & \text{if a center is to be located in } A_j(z^*) \\ 0 & \text{otherwise} . \end{cases}$$

Also, an m component vector $v_j(z^*)$ is defined for each j, where the $i^{th}$ component is denoted by $v_{ji}(z^*)$ and is defined by

$$v_{ji}(z^*) = \begin{cases} 1 & \text{if } i \in I_j(z^*) \\ 0 & \text{otherwise.} \end{cases}$$

The vector $v_j(z^*)$ is said to characterize the intersection of the diamonds $D(P_i,z^*)$ for $i \in I_j(z^*)$, since it specifies which diamonds intersect to determine $A_j(z^*)$.

Thus if $v_{ji}(z^*) = 1$, a center $X_j$ located in $A_j(z^*)$ would be a point in $D(P_i,z^*)$ for all $i \in I_j(z^*)$. For each point $P_i$ there must be at least one center $X_j$ such that $X_j \in D(P_i,z^*)$; this constraint is expressed by

$$\sum_{j=1}^{M(z^*)} v_{ji}(z^*)\ x_j \geq 1$$

for $1 \leq i \leq m$. Then the set covering problem determined by $z^*$ may be written as

$$\begin{aligned} &\min \sum_{j=1}^{M(z^*)} x_j \\ &\text{s.t.} \sum_{j=1}^{M(z^*)} v_{ji}(z^*)\ x_j \geq 1 \qquad 1 \leq i \leq m \qquad (4.2.3) \\ &\qquad x_j = 0,1, \qquad 1 \leq j \leq M(z^*). \end{aligned}$$

Example 4.2.1. $P_1 = (-.5,4)$, $P_2 = (-1,3)$, $P_3 = (0,3)$, $P_4 = (2,3.5)$; $w_i = 1$, $g_i = 0$ for $1 \leq i \leq 4$; $z^* = 1$. The diamonds $D(P_i,z^*)$ are illustrated in Figure 4.1. There are eight different nonempty intersections so that $M(z^*) = 8$, with $I_1 = \{1\}$, $I_2 = \{2\}$, $I_3 = \{3\}$, $I_4 = \{4\}$, $I_5 = \{1,2\}$, $I_6 = \{1,3\}$, $I_7 = \{2,3\}$, $I_8 = \{1,2,3\}$. The corresponding intersections $A_j(z^*)$ are given by $v_1(1) = [1,0,0,0]$, $v_2(1) = [0,1,0,0]$, $v_3(1) = [0,0,1,0]$, $v_4(1) = [0,0,0,1]$, $v_5(1) = [1,1,0,0]$, $v_6(1) = [1,0,1,0]$, $v_7 = [0,1,1,0,0]$, $v_8(1) = [1,1,1,0]$. Then the set covering problem may be written as

$$\min x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_8 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x_i = 0,1 \qquad 1 \leq i \leq 8 .$$

The optimal solution is $x_4 = x_8 = 1$ and all other $x_i = 0$. Thus any point in $A_4(1)$ and any point in $A_8(1)$ determine a 2-center for the diamonds $D(P_i,1)$.

Property 4.2.1. Suppose that for any fixed value of $p$, the optimal objective function value of problem (4.2.1) is $z^o$. Let $z^* = z^o$ and suppose $\underline{x}^o = (x_1^o,\ldots,x_{M(z^*)}^o)$ is an optimal solution to problem (4.2.3) with

$$\sum_{j=1}^{M(z^*)} x_j^o = p.$$

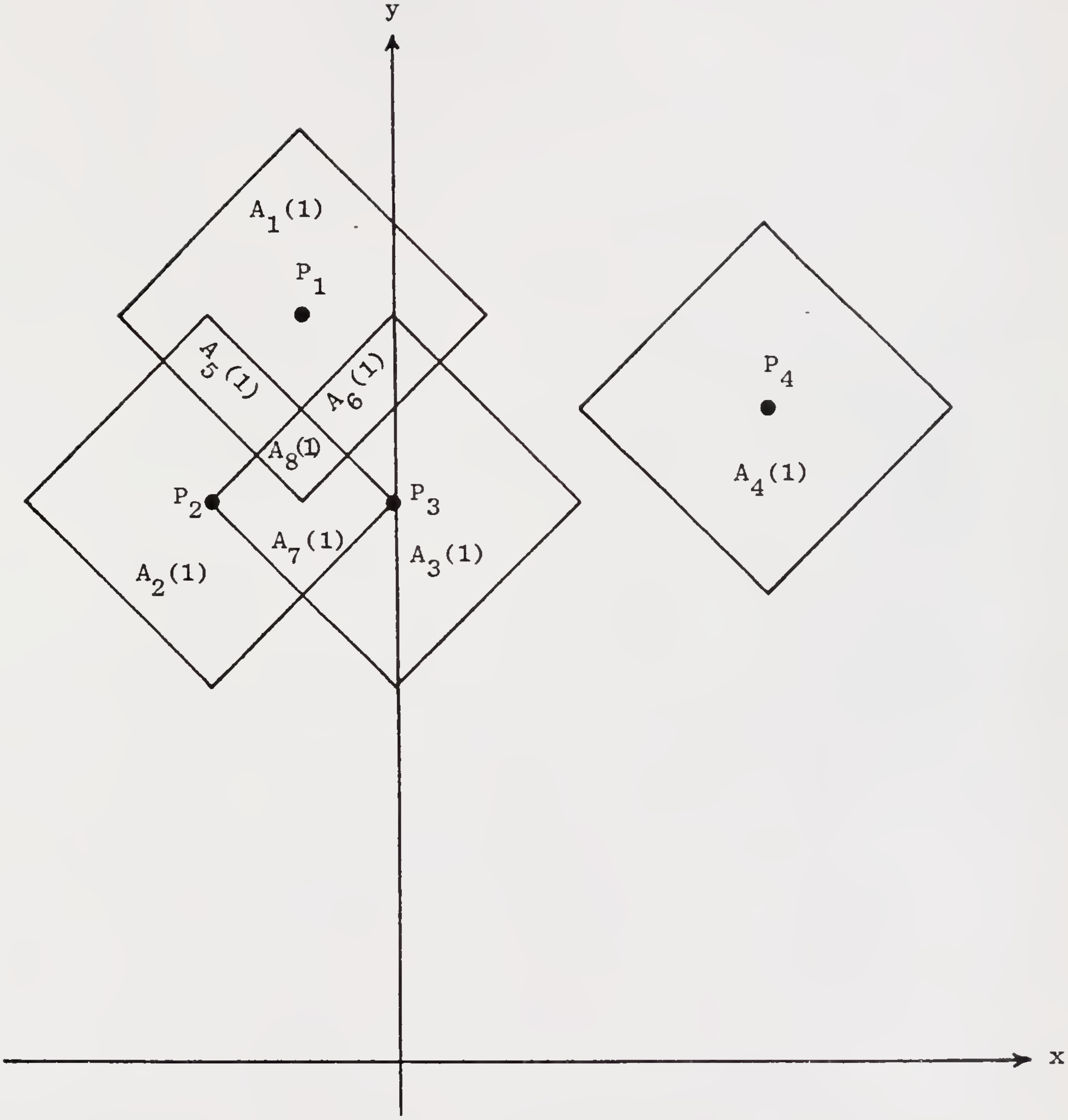

Figure 4.1 Diamonds $D(P_i,1)$

Define the p-center $\underline{X}_p^o = \{X_1^o, \ldots, X_p^o\}$ as follows: if $x_j = 1$, $X_j^o$ is any point in $A_j(z^*)$. Then $\underline{X}^o$ is an optimal solution to (4.2.1).

Proof: Since $\underline{x}^o$ is a solution to the set covering problem, for each i there is at least one $x_j = 1$ such that $v_{ji}(z^*)\, x_j = 1$. That is, for each i, there is at least one $A_j(z^*)$ such that for $X_j^o$ in $A_j(z^*)$

$$w_i \| X_j^o - P_i \| + g_i \leq z^* = z^o.$$

Therefore, for each i, $1 \leq i \leq m$

$$\min_{1 \leq j \leq p} w_i \| X_j^o - P_i \| + g_i \leq z^* = z^o$$

and the set $\underline{X}^o$ and the value $z^o$ is a feasible solution to (4.2.2) with p centers. Since $z^o$ is the minimum value of z, $\underline{X}^o$ is an optimal solution to (4.2.2) and hence to problem (4.2.1).

Corollary 4.2.1. Suppose the hypotheses of Property 4.2.1 hold but that the optimal solution $\underline{x}^o$ to (4.2.3) for $z^* = z^o$ has

$$\sum_{j=1}^{M(z^*)} x_j^o = r < p.$$

Then the set $\underline{X}^o = \{X_1^o, \ldots, X_r^o\} \cup \{X_{j_1}, \ldots, X_{j_{p-r}}\}$, where $X_j^o$ is defined in the statement of Property 4.2.1 and where $X_{j_k}$ is any point in any diamond $D(P_i, z^o)$, is an optimal solution to problem (4.2.1) for a fixed value of p.

Corollary 4.2.1 considers the case where $z^o$ is an optimal solution to a p-center problem, for a fixed value of p, but only r centers, $r < p$, are needed for an optimal solution. In this case the set covering problem determines only r centers. To obtain a p-center solution, any $p - r$ additional points $X_j$, as specified above, may be used in the p-center solution.

Hence, for a fixed value of p and an optimal objective function value $z^o$ to problem (4.2.1), Property 4.2.1 implies that an optimal solution to problem (4.2.1) may be determined using the optimal solution to problem (4.2.3). The procedure for solving (4.2.1) will be to determine candidates for the optimal objective function values, and for each candidate to solve the corresponding set covering problem.

## 4.3 Applications

Applications of the p-center problem may occur in contexts similar to the minimax location problems studied in the previous chapters, that is, in locating emergency service facilities in settings where travel distance is approximated by the rectilinear distance between facilities. However, in the location problems of previous chapters, a new facility is located so that it can respond to any existing facility; in a p-center problem a center is located so that it can respond only to those facilities nearest to it.

As an example [18], suppose there are m existing facilities and there are p policemen available for emergency call to the facilities, $(p < m)$. It is desired to locate the policemen so that for any emergency occurring at an existing facility, the maximum distance that the nearest policeman has to travel is minimized. Similar applications may occur in locating fire stations, ambulance centers, or other emergency services so that the maximum distance from any existing facility to the nearest service facility is minimized.

Applications may also be found in locating centers that emphasize accessibility. Suppose it is desired to locate p mail boxes in a city with respect to m population groups idealized as points $P_i$, $1 \leq i \leq m$. The mail box locations are to be determined so that the maximum distance anyone at some point $P_i$ has to travel to the nearest mail box is minimized.

In p-center problems, the constants $w_i$ and $g_i$ may have similar interpretations to those discussed in Chapter 2. That is, the $w_i$ may represent cost or some other constant per unit of distance, and the $g_i$ may represent fixed costs or constants.

## 4.4 Related Work

Hakimi [18] appears to have originally formulated the p-center minimax problem. His formulation was a p-center location problem on a graph, and no solution procedure was offered.

Christofides and Viola [3] consider the p-center problem for a graph and develop a solution procedure. The approach developed in this chapter for the p-center problem in the plane is similar to the approach taken on the graph problem by Christofides and Viola. On the edges of the graph, they determine regions that are defined by the intersections of "rays" of fixed length along the edges emanating from each node. A set covering problem is then solved to determine the minimum number of regions needed to cover the nodes of the graph. By incrementing the length of the rays, a sequence of set covering problems is solved. This approach may not determine the optimal objective function value for some p-center, but can iterate to an objective function value that is arbitrarily close to the optimal value. A set covering problem is solved at each iteration.

The rays that are used in [3] on the graph p-center problem are analogous to the diamonds that are used in Section 4.2. One result of this chapter is that for the p-center problem using rectilinear distances in the plane, the optimal objective function value is determined for each value of p.

Toregas et al. [27] consider a problem of locating centers in the plane. They assume that centers can only be located at existing facility locations. Sets of existing facilities are constructed that contain potential center locations within a specified time or distance of each existing facility. The problem of locating centers within these sets to respond to existing facilities is formulated as a set covering problem.

A p-center problem using general distances is considered by Hearn [19]. He develops an enumeration algorithm for solving the p-center problem for a specified p and gives extensive computational experience.

Total cost analogues to the p-center problem have been referred to as location-allocation problems. Cooper [ 4 ] develops heuristic procedures for these problems, assuming Euclidean distances, and gives additional references.

## 4.5 Problem Development

In this section, properties of problems (4.2.1) and (4.2.3) are developed that form a basis for a solution procedure for problem (4.2.1).

For each pair of points $P_i$ and $P_j$ for $1 \le i \le j \le m$, compute the constants $B_{ij}$ by

$$B_{ij} = \max \left\{ \frac{1}{w_i + w_j} \left[ w_i w_j \| P_i - P_j \|_1 + g_i w_j + g_j w_i \right], g_i, g_j \right\} .$$

The number of $B_{ij}$ computed is $\binom{m}{2} + m$, which is denoted by $m_2$. Next, order the $B_{ij}$ and set them equal to $t_k$ so that $t_k \le t_{k+1}$ for $1 \le k \le m_2 - 1$.

Property 4.5.1. Suppose p is a given integer value, $p \le m$. If $z^o$ is the optimal objective function value for the p-center problem, (4.2.1), then $z^o = t_k$ for some $1 \le k \le m_2$.

Proof: Let $\underline{X}_p^o = \{X_1^o, \ldots, X_p^o\}$ be an optimal solution to problem (4.2.1). Define the index sets $N_\nu$ for $1 \le \nu \le p$ by

$$N_\nu = \{i : X_\nu^o \in D(P_i, z^o)\} .$$

Consider the p one facility minimax location problems, written as constrained problems,

$$\begin{aligned} &\min z_\nu \\ &\text{s.t.} \quad w_i \| X_\nu - P_i \|_1 + g_i \le z_\nu \qquad i \in N_\nu \end{aligned} \tag{4.5.1}$$

for $1 \leq \nu \leq p$. By Property 2.8.2, the optimal objective function value of the $\nu^{th}$ problem equals some $B_{ij}$, say $B_{ij}^{\nu}$, which equals some $t_k$ for $1 \leq k \leq m_2$.

By definition of $N_\nu$, $X_\nu^o$ and $z^o$ are a feasible solution to the $\nu^{th}$ problem (4.5.1) so that $B_{ij}^{\nu} \leq z^o$ for $1 \leq \nu \leq p$. Let

$$z' = \max_{1 \leq \nu \leq p} \{B_{ij}^{\nu}\} .$$

Then $z' \leq z^o$.

Let $X'_\nu$ be an optimal solution to the $\nu^{th}$ problem (4.5.1) and consider the set $\underline{X}'_p = \{X'_1, \ldots, X'_p\}$. For each $i$, $1 \leq i \leq m$, $i \in N_\nu$ for some $\nu$, that is, for each $i$, there is some $X'_\nu$ that satisfies the expression

$$w_i \| X'_\nu - P_i \|_1 + g_i \leq z' .$$

Therefore, for each $i$, $1 \leq i \leq m$,

$$\min_{1 \leq \nu \leq p} w_i \| X'_\nu - P_i \|_1 + g_i \leq z'$$

and the set $\underline{X}_p$ and $z$ is a feasible solution to problem (4.2.2). Hence $z^o \leq z'$, so that $z^o = z' = t_k$ for some $1 \leq k \leq m_2$.

By Property 4.5.1, some value $t_k$, for $1 \leq k \leq m_2$, will be the optimal objective function value to (4.2.1) for any value of $p$. By Property 4.2.1, for each value of $t_k$ that is an optimal objective function value to problem (4.2.1) for a fixed value of $p$, the set covering problem (4.2.3) generated by $t_k$ determines an optimal p-center for problem (4.2.1). By solving problem (4.2.3) for each value of $t_k$ in increasing order of $t_k$, all p-center solutions, for $p = m, m-1, \ldots, 2$, to (4.2.1) are obtained. For each value of p, the smallest value of $t_k$ that gives a p-center solution to (4.2.2) is the optimal objective function value of (4.2.1).

## 4.6 Determining the Set Covering Problems

Attention is now directed toward deriving the column vectors for the set covering problem determined by the parameter $t_k$. A procedure is developed that determines a set of vectors, denoted by $V(t_k)$, that contains vectors $v_j(t_k)$ which characterize all the nonempty intersections of diamonds $D(P_i, t_k)$, $1 \leq i \leq m$, that are needed for the set covering problem generated by $t_k$. The vectors in $V(t_k)$ are used as the column vectors in the set covering problem determined by $t_k$. The problem of determining the set $V(t_k)$ of vectors $v_j(t_k)$, $1 \leq j \leq M(t_k)$, is decomposed into two subproblems; each subproblem determines a set of vectors, denoted by $V_x(t_k)$ and $V_y(t_k)$, respectively, that characterize all the intersections of intervals along the x and y axes.

For a given value of $t_k$, consider the diamonds $D(P_i, t_k)$ for $1 \leq i \leq m$. Applying the transformation T rotates each diamond onto a square with sides parallel to the rotated axes x and y, respectively. Define a square, in the rotated space, centered at $P'_i = (a'_i, b'_i)$ and of value $t'_k$ as the set

$$S(P'_i, t'_k) = \{X' \in E_2 : w_i \| X' - P'_i \|_\infty + g'_i \leq t'_k\}$$

where $g_i' = g_i/\sqrt{2}$ and $t_k' = t_k/\sqrt{2}$ , corresponding to the results in Chapter 2. Note that a square $S(P_i', t_k')$ has side length $2(t_k' - g_i')/w_i$.

Then each nonempty intersection of diamonds $D(P_i, t_k)$, for $i \in I_j(t_k)$, corresponds uniquely to the nonempty intersection of squares $S(P_i', t_k')$ for $i \in I_j(t_k)$. Consequently, the procedure actually determines vectors that characterize the nonempty intersection of squares in the rotated space; these vectors are denoted by $v_j(t_k')$ for $1 \leq j \leq M(t_k')$.

The problem of determining the vectors $v_j(t_k')$ may be decomposed into one-dimensional problems of determining vectors that characterize the intersection of intervals on the $x'$ and $y'$ axes, respectively, where the intervals are obtained by projecting the sides of the squares onto the respective axes. Explicitly, an interval centered at $a_i'$ of value $t^x$ is denoted by $I(a_i', t^x)$ and is defined by

$$I(a_i', t^x) = \{x' \in E_1 : w_i |x' - a_i'| + g_i' \leq t^x\} .$$

Notice that an interval $I(a_i', t^x)$ has length $2(t^x - g_i')/w_i$. Next, the problem of determining intersections of intervals is studied and a computational procedure is developed.

It will be shown that the vectors in $V(t_k)$ are determined from the vectors that characterize intersections of intervals on both axes.

Example 4.6.1. Consider Example 4.2.1 and apply the transformation T to obtain the squares as illustrated in Figure 4.2. The unique correspondence of intersections among diamonds and among squares may be

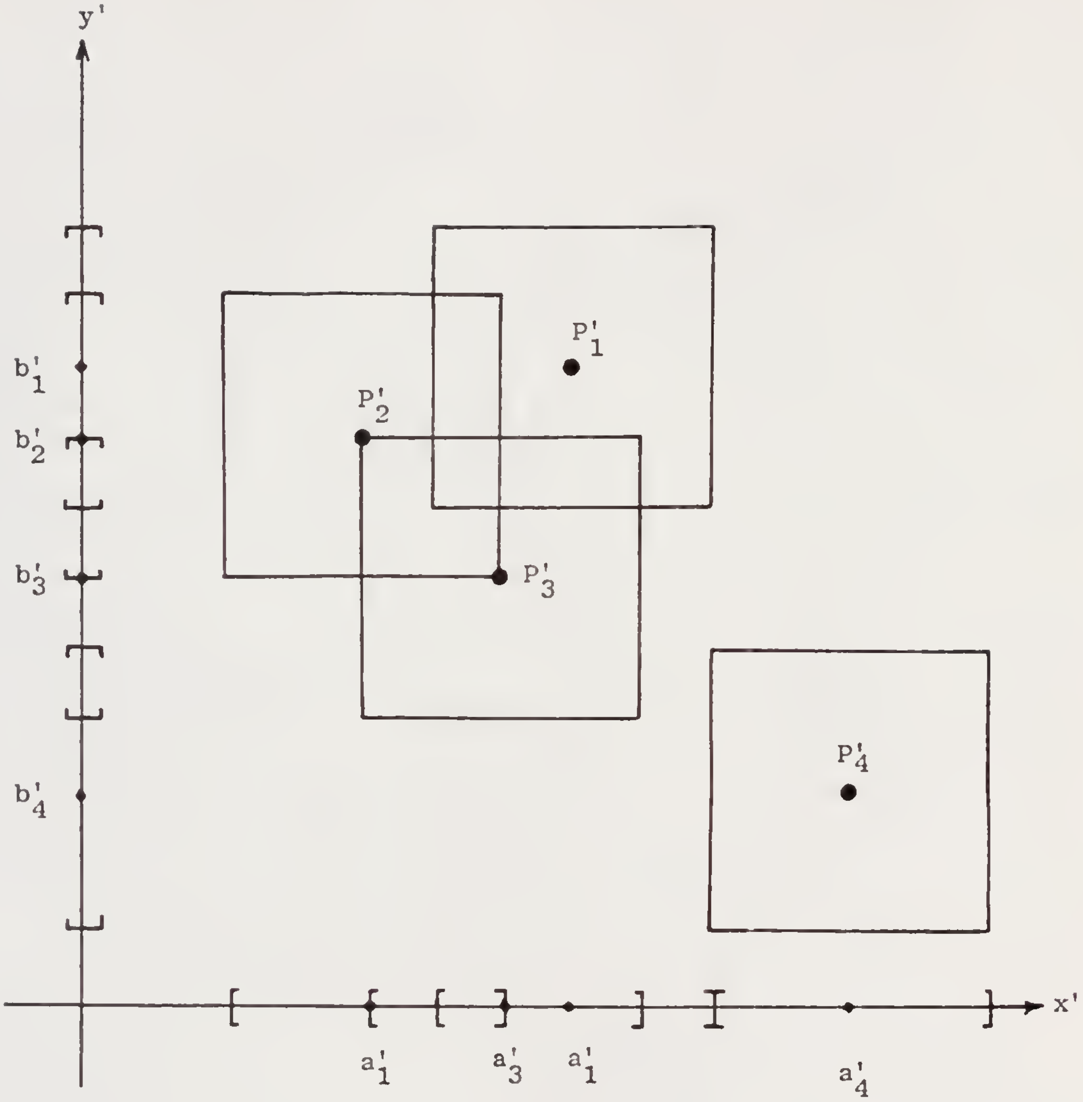

Figure 4.2 Squares $S(P'_i, 1/\sqrt{2})$

observed from Figures 4.1 and 4.2. Figure 4.2 also illustrates the intervals and their intersections, corresponding to sides of squares, along the x' and y' axes.

Now consider the problem of characterizing the intersections of intervals $I(a_i, t^x)$ by zero-one vectors. In order to define these vectors precisely, it is necessary to establish the following three lemmas.

For each pair of points $a'_i$ and $a'_j$, $1 \leq i \leq j \leq m$, compute the $m_2$ constants $b^x_{ij}$ defined by

$$b^x_{ij} = \max\left\{\frac{1}{w_i+w_j}\left[w_i w_j |a'_i - a'_j| + w_i g'_j + w_j g'_i\right],\ g'_i, g'_j\right\} .$$

Lemma 4.6.1. Suppose $i \neq j$, $b^x_{ij} > g'_i$ and $b^x_{ij} > g'_j$ with the points $a'_i$ and $a'_j$ named so that $a'_i < a'_j$. Define the point $x'_{ij}$ by

$$x'_{ij} = \frac{1}{w_i+w_y}\left[w_i a'_i + w_j a'_j + g'_j - g'_i\right] .$$

Then, $x'_{ij}$ is the unique point such that

$$w_i |a'_i - x'_{ij}| + g'_i = w_j |a'_j - x'_{ij}| + g'_j = b^x_{ij} .$$

Proof: The inequality $b^x_{ij} > g'_i$ and the definition of $b^x_{ij}$ imply

$$b^x_{ij} - g'_i = \frac{w_i}{w_x+w_y}\left[w_j |a'_i - a'_j| + g'_j - g'_i\right] > 0 . \qquad (4.6.1)$$

A similar expression with i and j interchanged and denoted by (4.6.2) results from the inequality $b^x_{ij} > g'_j$. The expressions (4.6.1) and (4.6.2) may be used to show that $a'_i < x'_{ij} < a'_j$ by showing that

$$x'_{ij} - a'_i = (b^x_{ij} - g'_i)/w_i > 0 \qquad (4.6.3)$$

and

$$a'_i - x'_{ij} = (b^x_{ij} - g'_j)/w_j > 0\ . \qquad (4.6.4)$$

But expressions (4.6.3) and (4.6.4) imply that

$$w_i|x'_{ij} - a'_i| + g'_i = w_j|x'_{ij} - a'_j| + g'_j = b^x_{ij}.$$

Then, these equations may be used to show that $x'_{ij}$ is the unique such point.

Lemma 4.6.2. Suppose $i \neq j$, and $b^x_{ij} = g'_i$ so that $b^x_{ij} \geq g'_j$. Define the point $x'_{ij} = a'_i$. Then,

$$w_i|x'_{ij} - a'_i| + g'_i = b^x_{ij} \quad \text{and} \quad w_j|x'_{ij} - a'_j| + g'_j \leq b^x_{ij}.$$

Proof: The assumptions imply that

$$g'_i \geq \frac{1}{w_i+w_j}\left[w_iw_j|a'_i - a'_j| + w_ig'_j + w_jg'_i\right].$$

This inequality implies $w_j|a'_i - a'_j| + g'_j \leq g'_i = b^x_{ij}$. Clearly, $w_i|a'_i - a'_i| + g'_i = g'_i = b^x_{ij}$. Hence the lemma is proved.

Lemma 4.6.3. Suppose $i \neq j$, and $b^x_{ij} = g'_j$, so that $b^x_{ij} \geq g'_i$. Define the point $x'_{ij} = a'_j$. Then,

$$w_i|x'_{ij} - a'_i| + g'_i \leq b^x_{ij} \quad \text{and} \quad w_j|x'_{ij} - a'_j| + g'_j = b^x_{ij}.$$

By the above three lemmas, if $i \neq j$, then the intersection $I(a'_i, b^x_{ij}) \cap I(a'_j, b^x_{ij})$ is nonempty since it contains $x'_{ij}$. Note also that if $i = j$, then $b^x_{ii} = g'_i$, and $x'_{ii} = a'_i$ is in the interval $I(a'_i, b^x_{ii})$. Hence, for any i and j, $1 \leq i \leq j \leq m$,

$$I(a'_i, b^x_{ij}) \cap I(a'_j, b^x_{ij}) \neq \emptyset.$$

For each $b^x_{ij}$, define a vector, denoted by $v(b^x_{ij})$, that characterizes the intersection $I(a'_i, b^x_{ij}) \cap I(a'_j, b^x_{ij})$, where the characterization is provided by having component k equal one if and only if $I(a'_k, b^x_{ij}) \cap I(a'_i, b^x_{ij}) \cap I(a'_j, b^x_{ij}) \neq \emptyset$. Clearly components i and j of $v(b^x_{ij})$ equal one. The next property provides a procedure for determining all the components of $v(b^x_{ij})$.

Property 4.6.1. The $k^{th}$ component of $v(b^x_{ij})$ equals one if and only if $b^x_{ij} \geq b^x_{ik}$ and $b^x_{ij} \geq b^x_{jk}$.

Proof: Suppose that $b^x_{ij} \geq b^x_{ik}$ and $b^x_{ij} \geq b^x_{jk}$. It must be shown that for some point x' in the intersection characterized by $v(b^x_{ij})$, $x' \in I(a_k, b^x_{ij})$.

Case A. $i = j$. Then, $b^x_{ii} = g'_i$ and the assumption gives $g'_i \geq b^x_{ik}$. Thus the following inequality holds:

$$\frac{1}{w_i + w_k}\left[w_i w_k \left|a'_i - a'_k\right| + w_i g'_k + w_k g'_i\right] \leq g'_i, \tag{4.6.5}$$

which implies $w_k \left|a'_i - a'_k\right| + g'_k \leq g'_i$ and $a'_i \in I(a'_k, g'_i)$, so that the $k^{th}$ component is one.

Case B. $i \neq j$. Then by Lemmas (4.6.1), (4.6.2) and (4.6.3), $I(a'_i, b^x_{ij}) \cap I(a'_j, b^x_{ij}) \neq \emptyset$. Four cases in terms of the $b^x_{ik}$ and $b^x_{jk}$ need to be considered.

1a: $b^x_{ik} > g'_i$ and $b^x_{ik} > g'_k$. Then $i \neq k$, and Lemma 4.6.1 implies $I(a'_i, b^x_{ik}) \cap I(a'_k, b^x_{ik}) \neq \emptyset$; since $b^x_{ik} \leq b^x_{ij}$,

$$I(a'_i, b^x_{ij}) \cap I(a'_k, b^x_{ij}) \neq \emptyset.$$

1b: $b^x_{jk} > g'_j$ and $b^x_{jk} > g'_k$. Then, similarly,

$$I(a'_j, b^x_{ij}) \cap I(a'_k, b^x_{ij}) \neq \emptyset .$$

2a: $b^x_{ik} \geq g'_i$, $b^x_{ik} = g'_k$. If $i \neq k$, Lemma (4.6.3) implies that $I(a'_i, b'_{ik}) \cap I(a'_k, b'_{ik}) \neq \emptyset$ so that $I(a'_i, b^x_{ij}) \cap I(a'_k, b^x_{ij}) \neq \emptyset$, since by assumption $b^x_{ik} \leq b^x_{ij}$. The same conclusion holds if $b^x_{ik} = g'_i$ and $b^x_{ik} \geq g'_k$ by Lemma (4.6.2). If $i = k$, $b^x_{ii} = g'_i$ and $I(a'_i, b^x_{ii}) \cap I(a'_i, b^x_{ii}) \neq \emptyset$.

2b: $b^x_{jk} \geq g'_j$, $b^x_{jk} = g'_k$, or $b^x_{jk} = g'_j$, $b^x_{jk} \geq g'_k$. Then analogous to Case 2a, $I(a'_j, b^x_{ij}) \cap I(a'_k, b^x_{ij}) \neq \emptyset$.

Therefore in any pairing of cases a and b the resulting non-empty intersections of intervals, combined with

$$I(a'_i, b^x_{ij}) \cap I(a'_j, b^x_{ij}) \neq \emptyset,$$

imply that

$$I(a'_i, b^x_{ij}) \cap I(a'_j, b^x_{ij}) \cap I(a'_k, b^x_{ij}) \neq \emptyset.$$

Hence there is an $x'$ in $I(a_i', b_{ij}^x) \cap I(a_j', b_{ij}^x)$ that is also in $I(a_k', b_{ij}^x)$.

Conversely, if the $k^{th}$ component of $v(b_{ij}^x)$ is one, then $w_k|x' - a_k'| + g_k' \leq b_{ij}^x$ for some $x'$ in $I(a_i', b_{ij}^x) \cap I(a_j', b_{ij}^x)$. Suppose

$$b_{ik}^x = \frac{1}{w_i + w_k}\left[w_i w_k |a_i' - a_k'| + w_i g_k' + w_k g_i'\right].$$

Then the triangle inequality gives

$$b_{ik}^x \leq \frac{1}{w_i + w_k}\left[w_i w_k |a_i' - x'| + w_k g_i' + w_i w_k |a_k' - x'| + w_i g_k'\right]$$

$$\leq \frac{1}{w_i + w_k}\left[w_k b_{ij}^x + w_i b_{ij}^x\right]$$

$$= b_{ij}^x.$$

If $b_{ik}^x = g_i'$, then by definition of $b_{ij}^x$, $g_i' \leq b_{ij}^x$. Finally, if $b_{ik}^x = g_k'$,

$$g_k' \leq w_k|x' - a_k'| + g_k' \leq b_{ij}^x.$$

A similar argument proves the second inequality, $b_{jk}^x \leq b_{ij}^x$, and the proof is complete.

For a given value of $b_{pq}^x$, the next property gives a method of determining all the vectors that characterize the collection of non-empty intersections of intervals $I(a_i', b_{pq}^x)$ for $1 \leq i \leq m$.

Property 4.6.2. Let $V_x^1(b_{pq}^x) = \{v(t_k^x) : t_k^x \le b_{pq}^x\}$ for a given value of $b_{pq}^x$. If $v(t_k^x) \in V_x^1(b_{pq}^x)$, than $v(t_k^x)$ characterizes a nonempty intersection of the intervals $I(a_i', b_{pq}^x)$. Conversely, each nonempty intersection of the intervals $I(a_i', b_{pq}^x)$ is characterized by a vector $v(t_k^x)$, for some $t_k^x \le b_{pq}^x$.

Proof: Suppose $v(t_k^x) \in V_x^1(b_{pq}^x)$. Since $t_k^x \le b_{pq}^x$, $I(a_i', t_k^x) \subset I(a_i', b_{pq}^x)$ for all i, so that any intersection of the intervals $I(a_i', t_k^x)$ for some subset of the i will be contained in the intersection of $I(a_i', b_{pq}^x)$ for the same i. Therefore, $v(t_k^x)$ characterizes a nonempty intersection of the intervals $I(a_i', b_{pq}^x)$, for $t_k^x \le b_{pq}^x$.

Now suppose G is a subset of the indices i, $1 \le i \le m$, such that

$$\bigcap_{i \in G} I(a_i', b_{pq}^x) \ne \emptyset \; .$$

Consider the points $a_i'$, $i \in G$, as fixed points in a one facility minimax location problem. Then the optimal objective function value is

$$b_{st}^x = \max_{i,j \in G} \{b_{ij}^x\} ,$$

and $b_{st}^x$ is the minimal value such that the intervals $I(a_i', b_{st})$ have nonempty intersection. Thus $b_{st}^x \le b_{pq}^x$ and $b_{st}^x = t_k^x$ for some k. By Property 4.6.1, $v(b_{st}^x)$ characterizes the nonempty intersection of $I(a_i', b_{pq}^x)$ for i in G.

Some of the vectors characterizing intersections of intervals may be redundant. For example, if the intersection corresponding to $v(t_{k_1}^x)$ contains the intersection corresponding to $v(t_{k_2}^x)$, then $v(t_{k_2}^x)$

is not needed. In this case, the vector $v(t^x_{k_1})$ has a one-component for each one-component of $v(t^x_{k_2})$.

Redundant vectors may be identified using the following definitions. A vector $v(t^x_{k_1})$ is said to dominate a vector $v(t^x_{k_2})$ if $v(t^x_{k_1}) \geq v(t^x_{k_2})$ in the vector sense [17]. Then a dominated vector $v(t^x_k)$ is redundant. An equivalent expression of dominance, and one that is useful computationally is given by the following definition. The Boolean product [3] of two zero-one vectors $v_1$ and $v_2$, denoted by $v_1 \circ v_2$, is the vector whose $k^{th}$ component is the product (logical and) of the $k^{th}$ components of $v_1$ and $v_2$, respectively. Then $v(t^x_{k_1})$ dominates $v(t^x_{k_2})$ if and only if $v(t^x_{k_1}) \circ v(t^x_{k_2}) = v(t^x_{k_2})$.

Let $V_x(b^x_{ij})$ be a subset of the vectors $V^1_x(b^x_{ij}) = \{v(t^x_k) : t^x_k \leq b^x_{ij}\}$, with the property that no vector in $V_x(b^x_{ij})$ is dominated by any other vector in $V_x(b^x_{ij})$. Then $V_x(b^x_{ij})$ is obtained from $V^1_x(b^x_{ij})$ by eliminating all the dominated vectors. The vectors in $V_x(b^x_{ij})$ still characterize the intersections of intervals $I(a_i, b^x_{ij})$.

An extension of the set $V_x(b^x_{ij})$ is now considered that has application later in the sequel.

Remark 4.6.1. Suppose t is any real number and consider the set $V_x(t^x_p)$ where $t^x_p$ is the largest value of $t^x_i$, $1 \leq i \leq m_2$ such that $t^x_p \leq t$, then $V_x(t^x_p) = V_x(t)$.

Proof: Since $t^x_p \leq t$, $I(a'_i, t^x_p) \subset I(a'_i, t)$ for $1 \leq i \leq m$, and any intersection of $I(a'_i, t^x_p)$ is contained in the respective intersection of $I(a'_i, t)$; thus $V_x(t^x_p) \subset V_x(t)$.

Suppose $v(t)$ is a vector in $V_x(t)$; let G be the set of indices i such that component i in $v(t)$ equals one. Suppose

$$b^x_{pq} = \max_{i,j \in G} \left\{ b^x_{ij} \right\} .$$

Then, $b^x_{pq} \leq t$ by the logic in the proof of Property 4.6.2. Therefore, $v(t)$ is in $V_x(t^x_p)$.

The procedure developed above may also be used to determine the set $V_y(t)$ of vectors characterizing the intersections of intervals $I(b'_i, t)$ on the y' axis for any real value of t.

The above development provides a method for characterizing the intersections of intervals of a specified value on either axis. Attention is now directed toward characterizing the intersections of squares using the sets of vectors $V_x(t^x_k)$ and $V_y(t^y_k)$ that characterize interval intersections in each axis.

Property 4.6.3. $v_j(t'_k)$ is a vector characterization of an intersection of squares in the rotated space if and only if there exist a vector $v(t^x_{k_1})$ in $V_x(t'_k)$ and a vector $v(t^y_{k_2})$ in $V_y(t'_k)$ such that

$$v_j(t'_k) = v(t^x_{k_1}) \circ v(t^y_{k_2}).$$

Proof: Suppose there exist vectors $v(t^x_{k_1})$ in $V_x(t'_k)$ and $v(t^y_{k_2})$ in $V_y(t'_k)$ whose Boolean product is $v_j(t'_k)$. Let G be the set of indices i such that component i equals one in both $v(t^x_{k_1})$ and $v(t^y_{k_2})$; consider the squares with center $(a'_i, b'_i)$ and with value $t'_k$ for i in G. Let x' be a point in the intersection characterized by $v(t^x_{k_1})$ and let y' be a point in the intersection characterized by $v(t^y_{k_2})$. Then for i in G,

$$w_i|a_i' - x'| + g_i' \leq t_{k_1}^x \leq t_k'$$

and $$w_i|b_i' - y'| + g_i' \leq t_{k_2}^y \leq t_k'.$$

Thus $$w_i\|(a_i',b_i') - (x',y')\|_\infty + g_i' \leq t_k'$$

and the point $(x',y')$ is in the intersection of the squares $S((a_i',b_i'),t_k')$ for i in G. That is, $v_j(t_k')$ characterizes an intersection of squares in the rotated space.

Conversely, suppose $v_j(t_k')$ is a characterization of an intersection of squares. Let the set of components i of $v_j(t_k')$ that equal one be denoted by G. Consider the intervals $I(a_i',t_p^x)$ and $I(b_i',t_q^y)$, for i in G and where $t_p^x$ and $t_q^y$ are the largest values of $t_k^x$ and $t_k^y$, respectively, such that $t_p^x \leq t_k'$ and $t_q^y \leq t_k'$. Suppose that the intervals $I(a_i',t_p^x)$ have empty intersection for i in G, then an argument similar to the last half of the proof of Remark 4.6.1 shows that the intervals $I(a_i',t_k')$ have empty intersection for i in G. But this implies that the squares $S((a_i',b_i'),t_k')$ have empty intersection. Therefore $\bigcap_{i\in G} I(a_i',t_p^x) \neq \emptyset$. A similar argument shows that $\bigcap_{i\in G} I(b_i',t_q^x) \neq \emptyset$.

Let $v(t_p^x)$ and $v(t_q^y)$ be the respective vector characterizations of these nonempty intersections of intervals. Then, since the two intersections immediately above are nonempty, the $i^{th}$ component of each equals one for i in G, which implies $v(t_p^x) \circ v(t_q^x) = v_j(t_k')$.

Thus for a fixed value of $t_k'$, the Boolean product of a pair of vectors, one from $V_x(t_k')$ and one from $V_y(t_k')$, generates a vector $v_j(t_k')$ that characterizes an intersection of squares of value $t_k'$, and hence of diamonds of value $t_k$. The set of all vectors generated by all such pairs characterize all intersections of diamonds with radius $t_k$. Suppose that all the dominated vectors in this set are eliminated. Then the resulting set of vectors, denoted by $V(t_k)$, is the smallest number of vectors that completely characterize all nonempty intersections of diamonds of value $t_k$. These vectors are the column vectors for the set covering problem generated by $t_k$.

## 4.7 A Solution Procedure

A procedure is now given for obtaining a solution to the p-center problem for any p, $p = m, m-1, \ldots, 2$. For $p = 1$, the problem is a one facility location problem which is studied in Chapter 2.

Note also that an m-center solution is immediately given by locating a center at each of the fixed facility locations $P_i$ for $1 \leq i \leq m$ and has an objective function value $z^o$ where $z^o = \max_{1 \leq i \leq m} \{g_i\}$.

The following algorithm determines a solution to the p-center problem for a given value of p, say $p_1$. It also determines all p-centers for $p_1 \leq p \leq m$.

### Algorithm

1. Compute and order the $b^x_{ij}$; set them equal to $t^x_k$ with $t^x_k \leq t^x_{k+1}$. Compute and order the $b^y_{ij}$; set them equal to $t^y_k$ with $t^y_k \leq t^y_{k+1}$. Set $b_{ij} = \max (b^x_{ij}, b^y_{ij})$. Order the $b_{ij}$ and set them equal to $t_k$ with $t_k \leq t_{k+1}$.

   Set $k = 1$, $p_k = m$.

2. Compute the sets $V_x(t'_k)$, $V_y(t'_k)$ and the set $V(t_k)$ from the Boolean product of vectors in $V_x(t'_k)$ and $V_y(t'_k)$, respectively. Solve the set covering problem using vectors in $V(t'_k)$ as column vectors. Suppose the optimal objective function value is $p_o$. If $p_o < p_k$, go to 3. If $p_o = p_k$, go to 4.

3. The value $t_k$ is the optimal objective function value for the p-center problem with $p_o \leq p < p_k$. If $p_o \leq p_1$ stop. Otherwise set $p_k = p_o$, $k = k+1$ and go to 2.

4. Set $k = k+1$, go to 2.

The algorithm is simply a restatement of the procedure developed in Sections 4.5 and 4.6 and therefore determines a p-center solution for the desired value $p_1$.

A set covering problem is solved for each value of $t_k$, unless $V(t_k) = V(t_{k+1})$. The maximum number of set covering problems needed is $m_2$ where $m_2 = \binom{m}{2} + m$. Thus as m gets large, large numbers of set covering problems are generated and the speed of the algorithm is largely dependent on the speed of solving set covering problems.

The algorithm was programmed for an IBM/65 at the University of Florida Computing Center. The set covering algorithm that is used in the program is a cutting plane algorithm developed and written by Ratliff [25].

Each vector is stored as a binary word. The generation of the vectors and the determination of dominating vectors is done using binary operations on words. Also some computational efficiency is gained by generating the vectors in the set $V(t_{k+1})$ using the vectors in the set $V(t_k)$.

Examples were computed using the programmed algorithm for $m = 5, 10, 15$, and 20 with the points $P_i$ sampled from random numbers with uniform distribution between 0 and 100, and the constants $w_i$ between 0 and 10; but with $g_i = 0$, for $1 \leq i \leq m$. From three to five examples of each size problem were computed. The cumulative execution time to generate all the set covering problems, and the cumulative execution time to solve all the set covering problems that are required to obtain a p-center solution were measured and recorded for each p, $2 \leq p \leq m-1$. The average of these times was taken over the example problems for each m and are presented as times A and B, respectively, in Tables 4.1 and 4.2. The total time required to obtain all p-centers is presented as time $C_1$.

In Tables 4.1 and 4.2, notice the initial time required to generate the first set covering problem for each m; but then notice that the accumulation of time to generate the succeeding set covering problems is fairly constant and a small proportion of time used to generate the initial problem.

For a given value of m, the time required to solve each set covering problem is fairly constant. The increase in the accumulation of time used in solving set covering problems to obtain succeeding p-centers is due to the increase in the number of set covering problems requiring a solution.

These computational results indicate that the algorithm does provide a computationally feasible method to solve p-center problems for all values of p when m is small.

Table 4.3 gives the input data and output data for the Example 4.2.1. Figures 4.3 and 4.4 illustrate the optimal p-center and the corresponding diamonds for p = 2 and 3, respectively.

## 4.8 Additional Observations

The algorithm of Section 4.7 may be initialized for any value of $t_k$, say $t_r$, by setting $k = r$ and computing $V(t_r)$ in step 2. Then to obtain the $p_1$-center, where $p_1$ is specified, computational savings may possibly be gained, particularly when $p_1$ is small, by starting the algorithm at a larger value of $t_k$. If the solution to the set covering problem generated by $t_r$ is p and $p \geq p_1$, then the algorithm would continue to iterate to the $p_1$-solution. If $p < p_1$, then $t_r$ was chosen too large, and the algorithm would have to be restarted at some $t_k < t_m$.

This discussion points out that considerable computational savings could be gained if bounds were known for the objective function value of the optimal $p_1$-center. Using these bounds, the procedure would require solutions only to set covering problems generated by $t_k$ that are within these bounds. No such bounds appear to be available presently, and additional research is needed to establish them.

Using the transformation T, a decomposition of the p-center problem into one-dimensional p-center subproblems may be obtained. Although the subproblems are not independent, solutions to the original problem may be obtained from solutions to some subproblems. Currently,

## TABLE 4.1

### AVERAGE EXECUTION TIMES FOR THE p-CENTER PROBLEM m = 20, 15

| | m = 20 | | | m = 15 | | |
|---|---|---|---|---|---|---|
| p | A | B | C | A | B | C |
| 19 | 20.074 | .677 | | | | |
| 18 | 20.441 | 1.273 | | | | |
| 17 | 20.815 | 2.209 | | | | |
| 16 | 21.151 | 2.677 | | | | |
| 15 | 21.514 | 3.577 | | | | |
| 14 | 21.854 | 4.108 | | 4.530 | .333 | |
| 13 | 22.120 | 4.601 | | 4.802 | .790 | |
| 12 | 22.564 | 5.489 | | 5.082 | 1.438 | |
| 11 | 22.910 | 6.256 | | 5.353 | 1.940 | |
| 10 | 23.365 | 8.067 | | 5.633 | 2.696 | |
| 9 | 23.780 | 9.278 | | 5.945 | 3.753 | |
| 8 | 24.262 | 11.278 | | 6.238 | 4.605 | |
| 7 | 24.841 | 14.944 | | 6.596 | 6.390 | |
| 6 | 25.418 | 17.802 | | 6.890 | 7.023 | |
| 5 | 26.060 | 22.201 | | 7.274 | 10.425 | |
| 4 | 26.848 | 29.254 | | 7.601 | 12.449 | |
| 3 | 27.792 | 38.831 | | 7.871 | 13.974 | |
| 2 | 28.808 | 52.457 | 81.265 | 8.156 | 16.835 | 24.991 |

A: average cumulative time in seconds to generate all $V(t_k)$ up to the p-center solution.

B: average cumulative time in seconds to solve all set covering problems up to the p-center solution.

C: total average time in seconds required to obtain all p-center solutions.

TABLE 4.2

AVERAGE EXECUTION TIMES FOR THE p-CENTER PROBLEM
$m = 10, 5$

| | m = 10 | | | m = 5 | | |
|---|---|---|---|---|---|---|
| p | A | B | C | A | B | C |
| 9 | 1.085 | .244 | | | | |
| 8 | 1.185 | .462 | | | | |
| 7 | 1.271 | .752 | | | | |
| 6 | 1.387 | .984 | | | | |
| 5 | 1.691 | 1.908 | | | | |
| 4 | 2.031 | 2.997 | | .070 | .070 | |
| 3 | 2.345 | 4.209 | | .176 | .189 | |
| 2 | 3.002 | 6.849 | 9.851 | .278 | .480 | .758 |

A: average cumulative time in seconds to generate all $V(t_k)$ up to the p-center solution.

B: average cumulative time in seconds to solve all set covering problems up to the p-center solution.

C: total average time in seconds to obtain all p-center solutions.

## TABLE 4.3

## DATA FOR EXAMPLE 4.2.1

Input

| $i$ | $a_i$ | $b_i$ | $w_i$ | $g_i$ |
|---|---|---|---|---|
| 1 | -0.5 | 4.0 | 1.0 | 0.0 |
| 2 | -1.0 | 3.0 | 1.0 | 0.0 |
| 3 | 0.0 | 3.0 | 1.0 | 0.0 |
| 4 | 2.0 | 3.5 | 1.0 | 0.0 |

Output

| $k$ | $t_k^x$ | $v(t_k^x)$ | $t_k^y$ | $v(t_k^y)$ | $t'_k$ |
|---|---|---|---|---|---|
| 1 | 0.177 | [0,1,0,0] | 0.177 | [0,0,1,1] | 0.354 |
| 2 | 0.354 | [0,1,0,0] | 0.354 | [0,1,1,0] | 0.530 |
| 3 | 0.530 | [0,1,1,1] | 0.530 | [1,0,1,0] | 0.530 |
| 4 | 0.707 | [1,0,0,1] | 0.530 | [0,1,1,1] | 0.884 |
| 5 | 0.884 | [1,1,0,1] | 0.884 | [1,1,1,0] | 1.061 |
| 6 | 1.237 | [1,1,1,1] | 1.061 | [1,1,1,1] | 1.237 |

$p = 3$

$$V(t'_1) = \left\{[0,0,0,1],\ [0,1,1,0],\ [1,0,0,0,]\right\}$$

$$\underline{X}_3^o = \left\{P_1 = (-0.5,4.0),\ P_4 = (2.0,3.5),\ X_3 = (-0.5,3.0)\right\}$$

$p = 2$

$$V(t'_2) = \left\{[0,0,0,1],\ [1,1,1,0]\right\}$$

$$\underline{X}_2^o = \left\{P_4 = (2.0,3.5),\ X_2 = (-0.5,3.25)\right\}$$

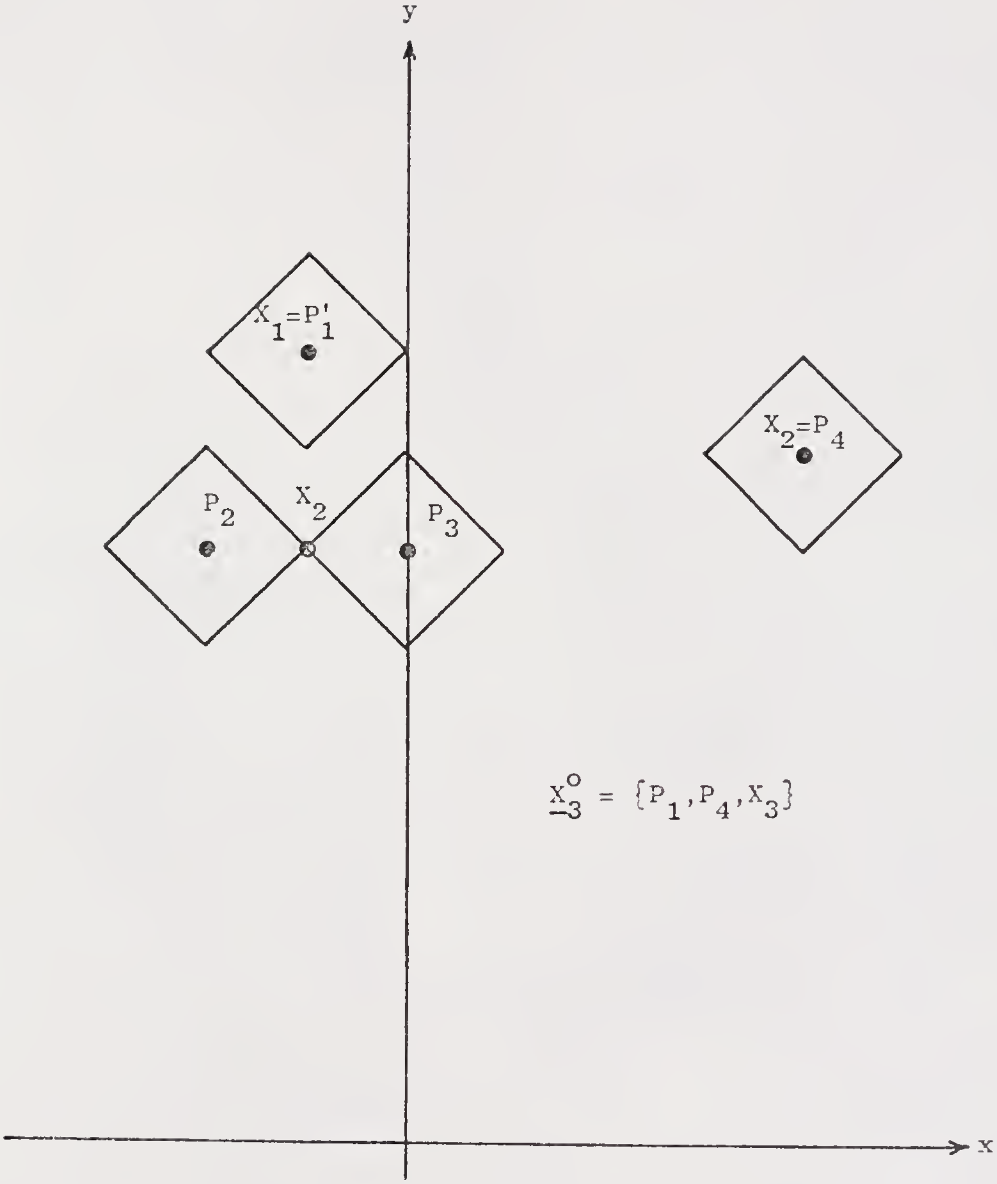

Figure 4.3 Optimal 3-Center for Example 4.2.1

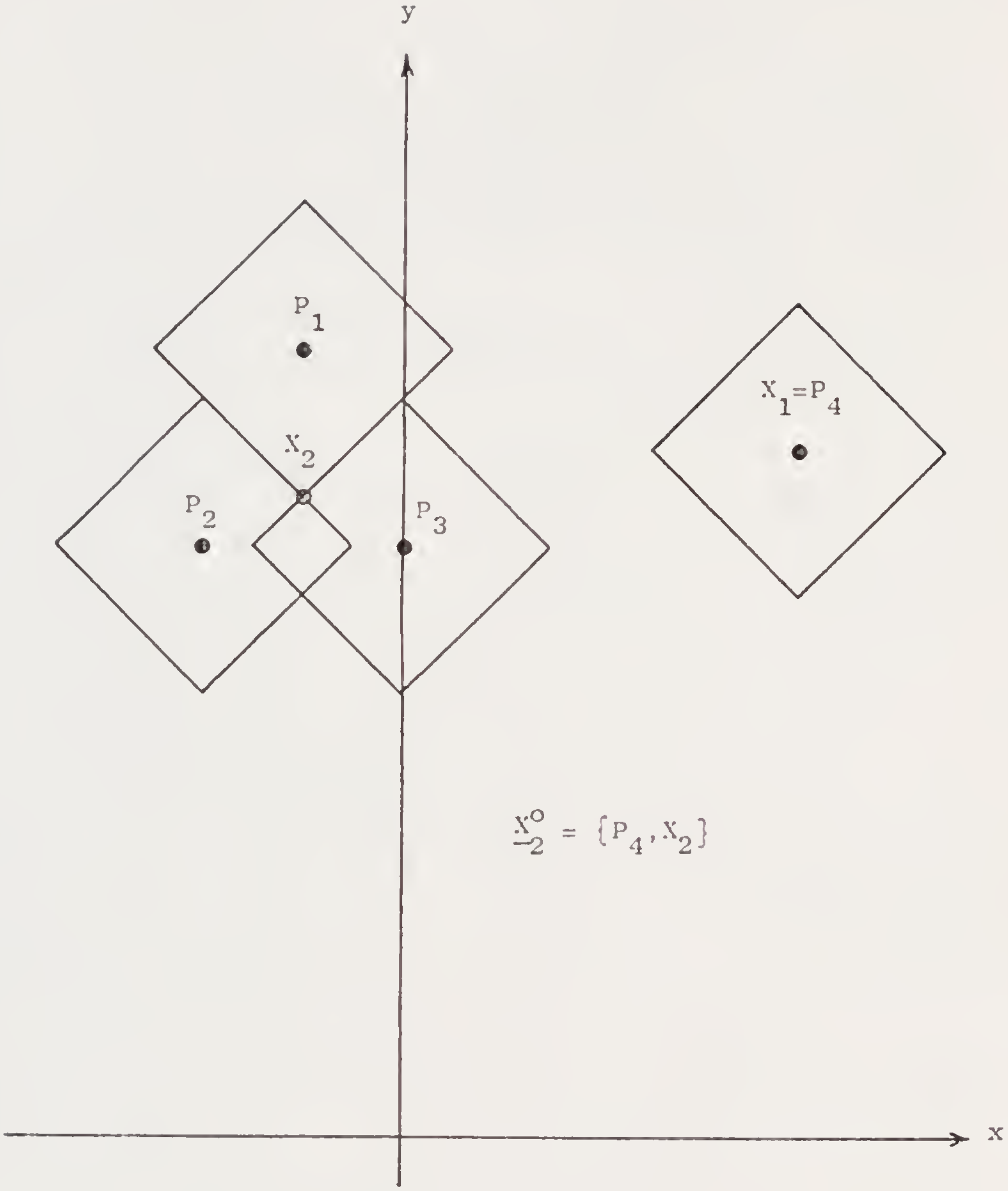

Figure 4.4 Optimal 2-Center for Example 4.2.1

solutions to the one-dimensional problems require the same procedure and computational effort as the original problem, that is, solving a sequence of set covering problems to determine the minimum number of intersections of intervals along an axis. However, the one-dimensional problems appear to have special structure, as do the one-dimensional minimax location problems studied in the previous chapters. Computational efficiencies might be gained if the subproblem structures allowed an improved solution procedure to the set covering problems.

# CHAPTER 5

# A ONE FACILITY PROBLEM WITH RESPECT TO A COMPACT SET

## 5.1 Introduction

In this chapter an extension of the one facility location problem studied in Chapter 2 is considered. The problem is to locate in the plane one new facility with respect to a compact set of existing facilities so as to minimize the maximum cost, where costs are linear functions of the rectilinear distance between new and existing facilities. The problem also includes upper bounds on the rectilinear distance from the new facility to each existing facility in the compact set.

The transformation T is applied to this problem and gives a decomposition into independent subproblems.

Some results are established which extend the work of Francis [13] for a one facility minimax location problem in $E_n$ with respect to a composite set and using more general distances then the rectilinear distance. These results are used to establish properties of the subproblems, which are analogous to those properties established in Chapter 2, and to obtain a solution procedure for the subproblems.

The optimal solutions to the subproblems determine the optimal solutions to the original problem.

## 5.2 Formulation

Let X denote a point in $E_2$ to be determined for the location of the new facility. Let S be a compact set in $E_2$ representing the location of existing facilities. For a point P = (a,b) in S, define w(P) to be a positive continuous function on S and define g(P) to be a continuous function on S. Define a function f(X) on $E_2$ by

$$f(X) = \max_{P \in S} \left\{ w(P) \| X - P \|_1 + g(P) \right\} .$$

Since w(P), g(P) and $\| X - P \|_1$ are continuous functions of P on S, the term inside the brackets is continuous. Thus the maximum over the compact set S exist and f(X) is well defined.

Let c(P) be a positive continuous function on S that is an upper bound on the rectilinear distance between X and P for each P in S.

The one facility problem to be studied in this chapter may be written as follows:

$$\min_{X \in E_2} \max_{P \in S} \left\{ w(P) \| X - P \|_1 + g(P) \right\} \tag{5.2.1}$$

$$\text{s.t.} \qquad \| X - P \|_1 \leq c(P), \qquad P \in S .$$

Notice that for the special case with S finite, problem (2.2.1) is obtained. Also, problem (5.2.1) without constraints, referred to as problem (5.2.2), is a special case of (5.2.1) obtained by letting c(P) be infinite on S.

## 5.3 Applications

The one facility location problem (5.2.1) may find applications in contexts of locating an emergency facility or a facility that is accessible to existing facilities, and in settings where the rectilinear distance approximates travel distance. The assumption that S is compact allows consideration of applications that may require the new facility to respond to an emergency at any point in some set or region. For example, an ambulance center might be expected to respond to emergencies that could occur at any point in a city district, not just at a finite number of points.

The function w(P) may have interpretations analogous to those considered in the previous chapter, for example, a cost function per unit of distance; the function g(P) also may have analogous interpretations as a cost function on the set S.

## 5.4 Related Work

The literature on problems of locating new facilities with respect to compact sets of existing facilities is not as extensive as the literature on location problems for a finite set of existing facilities. Francis [15] considers a special case of problem (5.2.2) with $w(P) = 1$ and $g(P) = 0$; this work appears to be the only work explicitly using rectilinear distances. Hearn [19] considers a one facility minimax problem using Euclidean distances with respect to a convex polyhedron and references related work for Euclidean distances.

The developments in Section 5.5 are extensions, to the case with S compact and $g(P) \neq 0$, of corresponding results obtained by Francis [13] for the one facility minimax problem with S finite.

## 5.5 Properties of a Minimax Problem

Results are established in this section that will be used in developing a solution procedure for problem (5.2.1), but the results also hold for more general one facility minimax problems. Hence for completeness, the results are established for the general case; special cases of these results will be referenced explicitly when they are needed in later sections.

For any norm defined on $E_n$ and for any point X in $E_n$, let $\|X\|$ denote the norm of X. Then a distance function, relative to a norm, may be defined on $E_n$ by writing the distance between any two points X and Y in $E_n$ as $\|X - Y\|$. It can be shown [12] that any norm is convex, and hence continuous on $E_n$, and that $\|X - Y\|$ is convex and continuous on $E_{2n}$.

To formulate the problem under consideration, let $\|X\|$ be a norm on $E_n$, let S be a compact set in $E_n$ and let w(P) and g(P) be functions on $E_n$, with w(P) positive and continuous on S and g(P) continuous on S. Then the problem of interest may be written as

$$\min_{X \epsilon E_n} \max_{P \epsilon S} \left\{ w(P) \| X - P \| + g(P) \right\} . \tag{5.5.1}$$

Notice that if the distance in (5.5.1) is the rectilinear distance and if n = 2, problem (5.2.2) is obtained.

Define the function F(X) by

$$F(X) = \max_{P \epsilon S} \left\{ w(P) \| X - P \| + g(P) \right\} .$$

The function F(X) is well defined by an argument similar to that in Section 5.2.

In each of the following properties, the special case obtained by assuming S is finite and g(P) = 0 on S corresponds to a similar property obtained by Francis [12].

First a lower bound for $F(X)$ is obtained. Define a real valued function, $B(P_\alpha,P_\beta)$, for all $(P_\alpha,P_\beta)$ in $S \times S \subset E_{2n}$ by

$$B(P_\alpha,P_\beta) = \max\left\{\frac{1}{w(P_\alpha)+w(P_\beta)}\Big[w(P_\alpha)w(P_\beta)\|P_\alpha - P_\beta\| + w(P_\alpha)g(P_\beta) + w(P_\beta)g(P_\alpha)\Big],\ g(P_\alpha),\ g(P_\beta)\right\} . \qquad (5.5.2)$$

Then $B(P_\alpha,P_\beta)$ is continuous on $S \times S$, since each component is continuous on $S \times S$. Also $S \times S$ is compact in $E_{2n}$ since $S$ is compact in $E_n$. Therefore there exist a point, say $(P_s,P_t)$ in $S \times S$ and a number, $B_{st}$, defined by

$$B_{st} = \max_{(P_\alpha,P_\beta)\in S\times S} \left\{B(P_\alpha,P_\beta)\right\} = B(P_s,P_t) .$$

Property 5.5.1. For all $X$ in $E_n$, $B_{st} \leq F(X)$.

Property 5.5.2. $B_{st} = F(X)$ if and only if

(1) $\|P_s - P_t\| = \|P_s - X\| + \|X - P_t\|$ ,

(2) $w(P_s)\|X - P_s\| + g(P_s) = w(P_t)\|X - P_t\| + g(P_t)$,

and (3) $w(P)\|X - P\| + g(P) \leq B_{st}$ for all $P$ in $S$, $P \neq P_s,P_t$.

The proofs of Properties 5.5.1 and 5.5.2 are direct extensions of the proofs of the analogous results in [12].

Property 5.5.3. There exists an $X_o$ in $E_n$ such that $F(X_o) \leq F(X)$ for all $X$ in $E_n$.

Proof: For each P in S, $w(P)\|X-P\|+g(P)$ is a convex function in X on $E_n$. Since the maximum of convex functions is convex,

$$\max_{P\epsilon S}\left\{w(P)\|X-P\|+g(P)\right\}=F(X)$$

is a convex function on $E_n$. Since F(X) is a convex function on $E_n$, and since $E_n$ is an open, convex set, F(X) is continuous on $E_n$. Therefore the set

$$K=\{X\in E_n : F(X)\leq k\}$$

for any real k is a closed set on $E_2$. K is also bounded so that it is compact. Thus, F(X) attains a minimum value for some X in K, and, as shown in [12], there exists an $X_o$ in $E_n$ such that $F(X_o)\leq F(X)$ for all X in $E_n$.

If $X_o$ is a minimum point of F(X), then by the definition of $B_{st}$, $F(X_o)\geq g(P)$ for all P in S. The following property concerns the case when $F(X_o)=g(P_s)$ for some $P_s$ in S.

Property 5.5.4. Suppose that $X_o$ is any point in $E_n$ such that $F(X_o)=g(P_s)$ for some point $P_s$ in S. Then $X_o=P_s$ is the unique minimum value of F(X).

Proof: Suppose $F(X_o)=g(P_s)$ for some $P_s$ in S, then

$$\begin{aligned} g(P_s) &= F(X_o)\\ &= \max_{P\epsilon S}\left\{w(P)\|X_o-P\|+g(P)\right\}\\ &\geq w(P_s)\|X_o-P_s\|+g(P_s)\end{aligned}$$

implies $0 \geq \|X_o - P_s\|$ so that $X_o = P_s$. Now suppose X is any point in $E_n$ with $X \neq P_s$, then

$$\begin{aligned} F(X) &\geq w(P_s)\|X - P_s\| + g(P_s) \\ &> g(P_s) = F(P_s) \ . \end{aligned}$$

Therefore $P_s$ is the unique minimum of F(X).

Under the assumption that $F(X_o) = g(P_s)$, it is possible that there is only one point in S, namely, $P_s$, such that

$$F(X_o) = w(P_s)\|X_o - P_s\| + g(P_s).$$

However, if $F(X_o) > g(P)$ for all P in S, then there are at least two points in S, namely, $P_s$ and $P_t$, such that

$$F(X_o) = w(P_s)\|X_o - P_s\| + g(P_s) = w(P_t)\|X_o - P_t\| + g(P_t).$$

This assertion is stated as a necessary condition for $X_o$ to be a minimum in the following property.

Property 5.5.5. Suppose S contains at least two points; let $X_o$ be such that $F(X_o) \leq F(X)$ for all X in $E_2$, and suppose $F(X_o) > g(P)$ for all P in S. Then there exist distinct points $P_s$ and $P_t$ such that

$$F(X_o) = w(P_s)\|X_o - P_s\| + g(P_s) = w(P_t)\|X_o - P_t\| + g(P_t).$$

Proof: By definition of F(X), there exists some point $P_s$ in S such that

$$F(X_o) = w(P_s)\|X_o - P_s\| + g(P_s) \geq w(P)\|X_o - P\| + g(P)$$

for all P in S, $P \neq P_s$. Assume that the assertion is not true, that is,

$$w(P_s)\|X_o - P_s\| + g(P_s) > w(P)\|X_o - P\| + g(P)$$

for all P in S, $P \neq P_s$.

Part I of the proof shows that under the above assumption there exists a neighborhood, $N(X_o,\sigma)$, such that

$$w(P_s)\|X - P_s\| + g(P_s) > w(P)\|X - P\| + g(P)$$

for all X in $N(X_o,\sigma)$ and for all P in S with $P \neq P_s$. Part II shows that there exists an X' in $N(X_o,\sigma)$ such that $F(X') < F(X_o)$ which contradicts the assumption and proves the theorem.

Part I: For each P in S, with $P \neq P_s$, define $\delta(P)$ by

$$\delta(P) = w(P_s)\|X_o - P_s\| + g(P_s) - w(P)\|X_o - P\| - g(P). \qquad (5.5.3)$$

Then $\delta(P)$ is positive for every $P \neq P_s$ in S by assumption.

For every $P \neq P_s$ in S, define

$$\epsilon_1(P) = \frac{\delta(P)}{4\|X_o - P\| + 1} \qquad (5.5.4)$$

so that $\epsilon_1(P)$ is positive and

$$\frac{\delta(P)}{2} - \epsilon_1(P)\|X_o - P\| > 0 .$$

Then, since w(P) is continuous on S, for each $P \neq P_s$ in S, there exists a $\rho_1(P) > 0$ such that

$$|w(P) - w(P_\alpha)| < \epsilon_1(P)$$

for all $P_\alpha$ in $S \cap N(P,\rho_1(P))$.

Also for every $P \neq P_s$ in S, define

$$\varepsilon_2(P) = \frac{1}{2}\left(\frac{\delta(P)}{2} - \varepsilon_1(P)\|X_o - P\|\right) \tag{5.5.5}$$

so that $\varepsilon_2(P)$ is positive and

$$\frac{\delta(P)}{2} - \varepsilon_1(P)\|X_o - P\| - \varepsilon_2(P) > 0 .$$

Then, since g(P) is continuous on S, for each $P \neq P_s$ in S there exists a $\rho_2(P) > 0$ such that

$$|g(P) - g(P_\alpha)| < \varepsilon_2(P)$$

for all $P_\alpha$ in $S \cap N(P, \rho_2(P))$.

Since w(P) is continuous on the compact set S, there exists a real number M such that $w(P) \leq M$ for all P in S.

Define $\rho(P)$ by

$$\rho(P) = \frac{1}{2M+1} \min\Big\{\rho_1(P),\ \rho_2(P),\ \delta(P)/2 - \varepsilon_1(P)\|X_o - P\| - \varepsilon_2(P)\Big\} . \tag{5.5.6}$$

Then $\rho(P)$ is positive and

$$\delta(P)/2 - \varepsilon_1(P)\|X_o - P\| - \varepsilon_2(P) - M\rho(P) > 0.$$

It also follows from the definition of $\rho(P)$ that $|w(P) - w(P_\alpha)| < \varepsilon_1(P)$ for all $P_\alpha$ in $S \cap N(P, \rho(P))$, and that $|g(P) - g(P_\alpha)| < \varepsilon_2(P)$ for all $P_\alpha$ in $S \cap N(P, \rho(P))$.

Now the collection of neighborhoods $\{N(P, \rho(P))\}$ for $P \neq P_s$ is an open cover for S, and since S is compact, there exists a finite open cover of S, denoted as $\{N(P_i, \rho(P_i))\}$ for $1 \leq i \leq n$. Then, for each i, the terms $\delta(P_i)$, $\epsilon_1(P_i)$, $\epsilon_2(P_i)$ and $\rho(P_i)$ are defined by (5.5.3), (5.5.4), (5.5.5), and (5.5.6), respectively. Furthermore, (5.5.6) implies that

$$\delta(P_i)/2 - \epsilon_1(P_i)\|X_o - P_i\| - \epsilon_2(P_i) - M\rho(P_i) > 0$$

for $1 \leq i \leq n$. For each i, define the function $f_i(X)$ by

$$f_i(X) = w(P_s)\|X - P_s\| + g(P_s) - w(P_i)\|X_o - P_i\| - g(P_i).$$

Then $f_i(X)$ is continuous, and $f_i(X_o) = \delta(P_i)$ for each i. Therefore, for each i, there exists a $\sigma_i' > 0$ such that $f_i(X) \geq \delta(P_i)/2 > 0$ for all X in $N(X_o, \sigma_i')$.

Define the positive number $\sigma_i$ by

$$\sigma_i = \frac{1}{2M+1} \min \Big\{\sigma_i', \ \delta(P_i)/2 - \epsilon_1(P_i)\|X_o - P_i\| - \epsilon_2(P_i) - M\rho(P_i)\Big\}. \tag{5.5.7}$$

Define $\beta_i$ for $1 \leq i \leq n$ by

$$\beta_i = \delta(P_i)/2 - \epsilon_1(P_i)\|X_o - P_i\| - \epsilon_2(P_i) - M\rho(P_i) - M\sigma_i,$$

and note that (5.5.7) implies that $\beta_i > 0$ for each i. Then it follows that $f_i(X) \geq \delta(P_i)/2 > 0$ for all X in $N(X_o, \sigma_i)$.

Now for all P in $N(P_i, \rho(P_i))$, for all X in $N(X_o, \sigma_i)$, and for each i, $1 \leq i \leq m$, it follows that

$$w(P_s)\|X - P_s\| + g(P_s) - w(P)\|X - P\| - g(P)$$
$$\geq w(P_s)\|X - P_s\| + g(P_s) - w(P)\|X - X_o\| - w(P)\|X_o - P_i\| - w(P)\|P_i - P\| - g(P)$$

(by the triangle inequality)

$\geq w(P_s)\| X - P_s\| + g(P_s) - M\sigma_i - M\rho\,(P_i) - w(P)\| X_o - P_i\| - g(P)$

(by the definitions of M, $\sigma_i$ and $\rho\,(P_i)$)

$$= w(P_s)\| X - P_s\| + g(P_s) - w(P_i)\| X_o - P_i\| - g(P_i) + w(P_i)\| X_o - P_i\| + g(P_i)$$
$$- w(P)\| X_o - P_i\| - g(P) - M\sigma_i - M\rho\,(P_i)$$

(by subtracting and adding $w(P_i)\| X_o - P_i\| + g(P_i)$)

$= f_i(X) + (w(P_i) - w(P))\| X_o - P_i\| + g(P_i) - g(P) - M\sigma_i - M\rho\,(P_i)$

(by the definition of $f_i(X)$)

$\geq \delta(P_i)/2 - \epsilon_1(P_i)\| X_o - P_i\| - \epsilon_2(P_i) - M\sigma_i - M\rho\,(P_i)$

(by the definitions of $\delta(P_i)$, $\epsilon_1(P_i)$ and $\epsilon_2(P_i)$)

$\geq \beta_i > 0$ .

Now let $\sigma = \min_{1\leq i\leq n}(\sigma_i)$ and let $\beta = \min_{1\leq i\leq n}(\beta_i)$.

Then for all X in $N(X_o,\sigma)$ and for all $P \neq P_s$ in S,

$$w(P_s)\| X - P_s\| + g(P_s) - w(P)\| X - P\| - g(P) \geq \beta > 0.$$

This proves Part I.

Part II: It will be shown that there exist some X' in $N(X_o,\sigma)$ with $X' \neq X_o$ amd $X' \neq P_s$ such that

$$w(P_s)\| X_o - P_s\| + g(P_s) > w(P_s)\| X' - P_s\| + g(P_s).$$

Let

$$\lambda = \frac{1}{2} \min \left\{ \frac{\sigma}{\| X_o - P_s\|_2} , 1 \right\}$$

so that $\lambda \| X_o - P_s \|_2 < \sigma$ and $\lambda < 1$, $0 < 1 - \lambda < 1$.

Choose $X' = (1 - \lambda)X_o + \lambda P_s$.

Since $f(X_o) > g(P_s)$, $w(P_s) \| X_o - P_s \| + g(P_s) > g(P_s)$ so that $X_o \neq P_s$.

Then $X' \neq P_s$ and $X' \neq X_o$. Furthermore,

$$\| X' - X_o \|_2 = \| \lambda P_s - \lambda X_o \|_2 = \lambda \| P_s - X_o \|_2 < \sigma$$

so that X' is in $N(X_o, \sigma)$. Also

$$\| X' - P_s \| = \| (1-\epsilon)X_o - (1-\epsilon)P_s \| = (1-\epsilon) \| X_o - P_s \| < \| X_o - P_s \| .$$

Thus

$$w(P_s) \| X' - P_s \| + g(P_s) < w(P_s) \| X_0 - P_s \| + g(P_s). \tag{5.5.8}$$

The result for Part I implies

$$w(P_s) \| X' - P_s \| + g(P_s) = \max_{P \in S} \left\{ w(P) \| X' - P \| + g(P) \right\} = F(X').$$

Therefore (5.5.8) implies $F(X') < F(X_o)$ and is a contradiction.

This proves Property 5.5.4.

## 5.6 The Transformation and Decomposition

The transformation T is applied to problem (5.2.1) to obtain a rotated problem and a decomposition that is analogous to the developments in Section 2.5. Let S' be the image of the set S under T. The following functions are defined on S': $W(P') = w(T^{-1}(P'))$, $G(P') = \frac{1}{\sqrt{2}} g(T^{-1}(P'))$, and $C(P') = \frac{1}{\sqrt{2}} c(T^{-1}(P'))$. Then for a point P in S and $P' = T(P) = (a', b')$ in S', it follows that $W(P') = w(P)$,

$G(P') = \frac{1}{\sqrt{2}}\, g(P)$ and $C(P') = \frac{1}{\sqrt{2}}\, c(P)$. Also each of these functions is continuous since each is the composition of the continuous functions $T^{-1}$ and $w(P)$, $g(P)$ or $c(P)$, respectively. With these definitions, the rotated problem is written as:

$$\begin{aligned} &\min_{X' \varepsilon E_2} \max_{P' \varepsilon S'} \left\{ W(P') \| X' - P' \|_\infty + G(P') \right\} \\ &\text{s.t.} \qquad \| X' - P' \|_\infty \leq C(P'), \quad P' \;\varepsilon\; S'. \end{aligned} \tag{5.6.1}$$

The equivalence of problems (5.2.1) and (5.6.1) is established by the following property which is analogous to Property 2.5.2.

Property 5.6.1. The point $X^o$ is an optimal solution to problem (5.2.1) if and only if $T(X^o)$ is an optimal solution to problem (5.6.1). If $z_o$ and $z'_o$ are the optimal objective function values for (5.2.1) and (5.6.1), respectively, then $z_o = z'_o \sqrt{2}$.

Two subproblems are now defined from the rotated problem. Recall that a point in S' is written $P' = (a', b')$.

$$\begin{aligned} &\min_{x' \varepsilon E_1} \max_{P' \varepsilon S'} \left\{ W(P') | x' - a' | + G(P') \right\} \\ &\text{s.t.} \qquad | x' - a' | \leq C(P'), \quad P' \;\varepsilon\; S', \end{aligned} \tag{5.6.2}$$

$$\begin{aligned} &\min_{y' \varepsilon E_1} \max_{P' \varepsilon S'} \left\{ W(P') | y' - b' | + G(P') \right\} \\ &\text{s.t.} \qquad | y' - b' | \leq C(P'), \quad P' \;\varepsilon\; S'. \end{aligned} \tag{5.6.3}$$

Each subproblem is a location problem in one dimension, but with respect to a compact set in $E_2$. The following property relates optimal solutions of the subproblems to optimal solutions to the rotated problem (5.6.1).

Property 5.6.2. If the point $x'_o$ is an optimal solution to problem (5.6.2) with optimal objective function value $z'_1$ and if $y'_o$ is an optimal solution to problem (5.6.3) with optimal objective function value $z'_2$, then $(x'_o, y'_o)$ is an optimal solution to problem (5.6.1) with optimal objective function value $z'_o = \max (z'_1, z'_2)$.

Thus the optimal solutions to the one facility problem (5.2.1) are obtained from the independent solutions of the subproblems (5.6.2) and (5.6.3).

The constraints of each of the problems (5.6.2) and (5.6.3) may be simplified. For problem (5.6.2) define

$$C'_1 = \max_{P' \epsilon S'} \left\{ a' - C(P') \right\}$$

$$C'_2 = \min_{P' \epsilon S'} \left\{ a' + C(P') \right\}$$

Then problem (5.6.2) is equivalent to

$$\begin{aligned} &\min_{x' \epsilon E_1} \ \max_{P' \epsilon S'} \left\{ W(P') |a' - x'| + G(P) \right\} \\ &\text{s.t.} \qquad C'_1 \leq x' \leq C'_2 \ . \end{aligned} \qquad (5.6.4)$$

Similarly, for problem (5.6.3) define

$$D'_1 = \max_{P' \epsilon S'} \left\{ b' - C(P') \right\}$$

$$D'_2 = \min_{P' \epsilon S'} \left\{ b' + C(P') \right\}$$

so that problem (5.6.3) is equivalent to

$$\begin{aligned} &\min_{y' \in E_1} \ \underset{P' \in S'}{\text{man}} \left\{ W(P')|b' - y'| + G(P') \right\} \\ &\text{s.t.} \qquad D'_1 \leq y' \leq D'_2. \end{aligned} \tag{5.6.5}$$

Problems (5.6.4) and (5.6.5) have identical structure so that it suffices to develop properties and solution procedures only for problem (5.6.4).

## 5.7 Properties of the Subproblems

The subproblems (5.6.4) and (5.6.5) written without constraints are:

$$\min_{x' \in E_1} \ \max_{P' \in S'} \left\{ W(P')|a' - x'| + G(P') \right\} , \tag{5.7.1}$$

$$\min_{y' \in E_1} \ \max_{P' \in S'} \left\{ W(P')|b' - y'| + G(P') \right\} . \tag{5.7.2}$$

Analogous to the development in Chapter 2, the optimal solutions to the constrained subproblems may be obtained from the optimal solutions to the unconstrained subproblems. Since problems (5.7.1) and (5.7.2) have identical structure, only problem (5.7.1) is considered explicitly.

Properties are now developed for problem (5.7.1). These properties are analogous to those properties presented in Section 5.5 for problem (5.5.1). However, problem (5.7.1) cannot be considered as a special case of problem (5.5.1), since (5.7.1) is a problem in one dimension with respect to a set S' in $E_2$. The properties of (5.7.1) are presented below as remarks; they may be proved in an analogous fashion to the corresponding properties in Section 5.5.

Define the function f(x') on $E_1$ by

$$f(x') = \max_{P' \in S'} \left\{ W(P')|x' - a'| + G(P') \right\} .$$

It is assumed that there exist two points $(a'_1, b'_1)$ and $(a'_2, b'_2)$ in S' with $a'_1 \neq a'_2$; otherwise $x' = a'_1$ is the optimal solution to (5.7.1) with $f(x') = \max_{P' \in S} G(P')$.

For $P'_\alpha = (a',b')$ and $P'_\beta(c',d')$ in $S'$ define

$$b(P'_\alpha,P'_\beta) = \max\left\{\frac{1}{W(P'_\alpha)+W(P'_\beta)}\Big[W(P'_\alpha)W(P'_\beta)|a' - c'| + W(P'_\alpha)G(P'_\beta) + W(P'_\beta)G(P'_\alpha)\Big], G(P'_\alpha), G(P'_\beta)\right\}$$

and $$b_{st} = b(P',P') = \max_{(P'_\alpha,P'_\beta)\in S\times S}\left\{b(P'_\alpha,P'_\beta)\right\} . \qquad (5.7.3)$$

Remark 5.7.1. For all $x'$ in $E_1$, $b_{st} \leq g(x')$.

Remark 5.7.2. $b_{st} = f(x')$ if and only if (1) $|a'_s - c'_t| = |a'_s - x'| + |x' - c'_t|$, (2) $W(P'_s)|x' - a'_s| + G(P'_s) = W(P'_t)|x' - c'_t| + G(P'_t)$, (3) $W(P')|x' - a'| + G(P') \leq b_{st}$ for all $P'$ in $S'$, $P' \neq P'_s, P'_t$.

Remark 5.7.3. There exists an $x'_o$ in $E_1$ such that $f(x'_o) \leq f(x')$ for all $x'$ in $E_1$.

Remark 5.7.4. Let $x'_o$ be such that $f(x'_o) \leq f(x)$ for all $x'$ in $E_1$, and $f(x'_o) > G(P')$ for all $P'$ in $S'$. Then there exist distinct points $P'_s$ and $P'_t$ such that

$$f(x'_o) = W(P'_s)|x'_o - a'_s| + G(P'_s) = W(P'_t)|x'_o - c'_t| + G(P'_t) .$$

Additional properties are now developed for problem (5.7.1). Analogous to the unconstrained one facility subproblem in Chapter 2, the lower bound $b_{st}$ is always attained at optimality for problem (5.7.1).

Property 5.7.1. Let $b_{pq}$ be the lower bound defined by (5.7.3) and let $z'_o$ be the optimal objective function value of (5.7.1). Then $z'_o = b_{pq}$.

Proof: Suppose $x'_o$ is an optimal solution to problem (5.7.1). If $f(x'_o) = z'_o = G(P'_s)$ for some $P'_s$ in $S'$, then by Remark 5.7.1 $b_{pq} \leq G(P'_s)$, but by taking $s = t$, $b_{ss} = G(P'_s)$. Since $G(P'_s) \geq b_{ss} = G(P'_s)$, $b_{pq} = b_{ss}$.

If $f(x'_o) > G(P')$ for all $P'$ in $S'$, then by Remark 5.7.4, there exist distinct points $P'_s$ and $P'_t$ in $S'$ such that

$$z'_o = W(P'_s)\left|x'_o - a'_s\right| + G(P'_s) = W(P'_t)\left|x'_o - c'_t\right| + G(P'_t)$$

which implies that $z'_o$ is determined by one of four possibilities.

(1) $x'_o - a'_s = (z'_o - G(P'_s))/W(P'_s)$ and $x'_o - c'_t = (z'_o - G(P'_t))/W(P'_t)$

(2) $x'_o - a'_s = (z'_o - G(P'_s))/W(P'_s)$ and $-x'_o + c'_t = (z'_o - G(P'_t))/W(P'_t)$

(3) $-x'_o + a'_s = (z'_o - G(P'_s))/W(P'_s)$ and $x'_o - c'_t = (z'_o - G(P'_t))/W(P'_t)$

(4) $-x'_o + a'_s = (z'_o - G(P'_s))/W(P'_s)$ and $-x'_o + c'_t = (z'_o - G(P'_t))/W(P'_t)$.

In each of the four cases eliminate $x'_o$ and solve for $z'_o$. Cases (1) and (4) imply that $z'_o$ is either undefined, with denominator $W(P'_s) - W(P'_t)$, or gives a value of $z'_o$ greater than the value resulting from cases (2) and (3). Since $z'_o$ is the minimum value, only cases (2) and (3) need be considered. Case (2) or (3) yields

$$z'_o = \frac{1}{W(P'_s)+W(P'_t)}\left[W(P'_s)W(P'_t)\ (\pm 1)\ (a'_s - c'_t) + W(P'_s)G(P'_t) + W(P'_t)G(P'_s)\right],$$

respectively. Since $\left|a'_s - c'_t\right| = \max\ (\pm(a'_s - c'_t))$,

$$z'_o \leq \frac{1}{W(P'_s)+W(P'_t)}\left[W(P'_s)W(P'_t)\left|a'_s - c'_t\right| + W(P'_s)G(P'_t) + W(P'_t)G(P'_s)\right]. \tag{5.7.4}$$

Recall that $z'_o > G(P')$ for all $P'$ in $S'$ so that $b_{pq}$ is the maximum of expressions like the right-hand side of (5.7.4); therefore, $b_{pq} \geq z'_o$, but Remark 5.7.1 gives $b_{pq} \leq z'_o$ so that $b_{pq} = z'_o$.

The next property of problem (5.7.1) corresponds to a similar result for problem (2.6.1) in Chapter 2.

Property 5.7.2. Problem (5.7.1) has a unique optimal solution.

Proof: Suppose problem (5.7.1) has optimal objective function value $z'_o$. If $z'_o = G(P')$ for some $P'$ in $S'$, then the discussion of this case in Section 5.5 shows that problem (5.7.1) has a unique optimal solution.

If $z'_o > G(P')$ for all $P'$ in $S'$, suppose that $x'_1$ and $x'_2$ are both optimal to (5.7.1) with $x'_1 \neq x'_2$. Then, by Remark 5.7.4, there exist $P'_1$ and $P'_2$ such that

$$z'_o = W(P'_1)|x'_1 - a'_1| + G(P'_1) = W(P'_2)|x'_2 - a'_2| + G(P'_2).$$

Remark 5.7.4 also allows the assumption that $P'_1 \neq P'_2$. Since $f(x')$ is a convex function, $x' = \lambda x'_1 + (1-\lambda)x'_2$ for $0 < \lambda < 1$, is also an optimal solution to problem (5.7.1). By the definition of $f(x')$, there exists some point $P'_3$ in $S'$ such that

$$z'_o = W(P'_3)|x' - a'_3| + G(P'_3)$$

which implies

$$\begin{aligned} z'_o &= W(P'_3)|\lambda(a'_3 - x'_1) + (1-\lambda)(a'_3 - x'_2)| + G(P'_3) \\ &\leq \lambda\Big[W(P'_3)|a'_3 - x'_1| + G(P'_3)\Big] + (1-\lambda)\Big[W(P'_3)|a'_3 - x'_2| + G(P'_3)\Big]. \qquad (5.7.5) \end{aligned}$$

Now observe that by definition of $f(x_1')$ and $f(x_2')$

$$W(P_3')\,|a_3' - x_1'| + G(P_3') \leq W(P_1')\,|a_1' - x_1'| + G(P_1') = z_o' \tag{5.7.6}$$

and

$$W(P_3')\,|a_3' - x_2'| + G(P_3') \leq W(P_2')\,|a_2' - x_2'| + G(P_2') = z_o' \;. \tag{5.7.7}$$

If strict inequality holds in either (5.7.6) or (5.7.7), then substituting into expression (5.7.5) gives

$$z_o' < \lambda z_o' + (1-\lambda) z_o' = z_o'$$

so that equality must hold in both (5.7.6) and (5.7.7). But equality in both expressions implies that

$$W(P_3')\,|a_3' - x_1'| + G(P_3') = W(P_3')\,|a_3' - x'| + G(P_3')$$

and

$$W(P_3')\,|a_3' - x_2'| + G(P_3') = W(P_3')\,|a_3' - x'| + G(P_3') .$$

These equations imply that either $x_1 = x_2$, a contradiction to the assumption, or that $x' = x_1'$ or $x' = x_2'$, a contradiction, since $0 < \lambda < 1$. Therefore problem (5.7.1) has a unique optimal solution.

The next result gives an explicit expression for the unique optimal solution to problem (5.7.1) in terms of the bound $b_{st}$.

Property 5.7.3. Suppose $b_{st} = b(P_s', P_t')$. If $b_{st} = G(P_s')$ define $x_{st}' = a_s'$; if $b_{st} = G(P_t')$, define $x_{st}' = a_t'$, and if $b_{st} > G(P_s')$ and $b_{st} > G(P_t')$, and for $a_s' < a_t'$, define

$$x'_{st} = \frac{1}{W(P'_s)+W(P'_t)}\left[W(P'_s)a'_s + W(P'_t)a'_t + G(P'_t) - G(P'_s)\right].$$

Then in any of the above cases, $x'_{st}$ is the optimal solution to (5.6.1).

Proof: Suppose $b_{st} = G(P'_s)$. Then by Property 5.7.1, if $x'_o$ is an optimal solution to (5.6.1), $f(x'_o) = b_{st} = G(P'_s)$ and by Property 5.5.2, $x'_o = a'_s$ is the unique optimal solution to (5.6.1). Hence $x'_{st} = a'_s$ is the unique optimal solution to (5.7.1).

Suppose $b_{st} = G(P'_t)$, then by a similar argument, $x'_{st} = a'_t$ is the unique optimal solution to (5.7.1).

Suppose $b_{st} > G(P'_s)$ and $b_{st} > G(P'_t)$. Then a development analogous to the proof of Lemma 4.6.1 shows that

$$W(P'_s)\left|a'_s - x'_{st}\right| + G(P'_s) = W(P'_t)\left|a'_t - x'_{st}\right| + G(P'_t) = b_{st},$$

and that $x'_{st}$ is the unique point satisfying these equalities. Also, analogous to the proof of Lemma 4.6.1, $a'_s < x'_{st} < a'_t$.

It remains to be shown that $f(x'_{st}) = b_{st}$. For any other point, say $P'_r$ in $S'$, $b_{st} \geq b_{sr}$ and $b_{st} \geq b_{rt}$, since $b_{st}$ is the maximum of all such expressions. Then a proof, analogous to the proof of Property 4.5.1 shows that

$$W(P'_r)\left|x'_{st} - a'_r\right| + G(P'_r) \leq b_{st}.$$

Therefore the conditions of Remark 5.7.2 are satisfied so that $f(x'_{st}) = b_{st}$ and $x'_{st}$ is the optimal solution to (5.7.1).

The preceding remarks and properties in this section indicate a procedure for determining the unique optimal solution to problem (5.7.1).

Recall however, that it is desired to obtain an optimal solution to the subproblem (5.6.2), which is equivalent to problem (5.6.4). If $x'_o$ is the optimal solution to problem (5.7.1) and if $x'_o$ is feasible to (5.6.4), then $x'_o$ is the optimal solution to (5.6.4). The case where $x'_o$ is infeasible to (5.6.4) is considered in the next property.

Property 5.7.4. If $x'_o$ is an optimal solution to (5.7.1) and $x'_o$ is infeasible to (5.6.4), then the optimal solution to (5.6.4) is either $C'_1$ or $C'_2$. Specifically, if $x'_o < C'_1$, $C'_1$ is the unique optimal solution to (5.6.4) or if $x'_o > C'_2$, $C'_2$ is the unique optimal solution to (5.6.4).

The proof of Property 5.7.4 follows directly from the proof of Property 2.6.1.

## 5.8 Solution Procedure

The properties developed in the preceding section for the subproblem (5.5.4) and the analogous properties that exist for subproblem (5.5.5) are used to determine a solution procedure for both subproblems. A procedure is also given for finding all the alternative solutions to the one facility problem in two dimensions, (5.2.1), using the solutions to the subproblems. This procedure is completely analogous to the procedure for the one facility problem in Chapter 2.

Subproblem (5.6.2) is solved by computing the lower bound from expression (5.7.3) and determining the value $x'_o$ using Property 5.7.3. If $x'_o$ is feasible to problem (5.6.4), then $x'_o$ is the unique optimal solution and $b_{st} = z'_1$ is the optimal objective function value.

If $x'_o$ is not feasible, the unique optimal solution is $x'_o = C'_1$ or $x'_o = C'_2$ as determined by Property 5.7.4, and

$$z'_1 = \max_{P' \varepsilon S'} \left\{ W(P') \left| a' - C'_j \right| + G(P') \right\}$$

for j = 1 or 2, respectively.

Similarly, subproblem (5.6.3) is solved by computing the lower bound $b_{st}$ from expression (5.7.3) but with b' and d' replacing a' and c', respectively. Then $y'_o$ is determined from $b_{st}$ using analogous results to those given in Property 5.7.3. If $y'_o$ is feasible to (5.6.3), then it is the unique optimal solution and $b_{st} = z'_2$ is the optimal objective function value; otherwise, $y'_o = D'_1$ or $y'_o = D'_2$ and

$$z'_2 = \max_{P' \varepsilon S'} \left\{ W(P') \left| D'_j - b' \right| + G(P') \right\}$$

for j = 1 or 2, respectively.

By Property 5.6.2, the optimal objective function value, $z'_o$, to the rotated problem (5.6.1) is given by $z'_o = \max (z'_1, z'_2)$ where $z'_1$ and $z'_2$ are the optimal objective function values to (5.6.2) and (5.6.3), respectively.

The point $(x'_o, y'_o)$ is one optimal solution to problem (5.6.1) by Property 5.6.2. Three cases need to be considered to determine all alternative solutions.

Case 1: $z'_o = z'_1 = z'_2$. Then $(x'_o, y'_o)$ is the unique optimal solution to (5.6.1).

Case 2: $z'_o = z'_1 > z'_2$. Then with a development similar to that in Section 2.7, the points $(x'_o, y')$ on the line segment determined by $(x'_o, y'_1)$ and $(x'_o, y'_2)$ are all optimal solutions to (5.6.1) where

$$y'_1 = \max\left[D'_1, \max_{P' \in S'} \left\{b' - (z'_o - G(P'))/W(P')\right\}\right]$$

$$y'_2 = \min\left[D'_2, \min_{P' \in S'} \left\{b' + (z'_o - G(P'))/W(P')\right\}\right].$$

Case 3: $z'_o = z'_2 > z'_1$. Then determine

$$x'_1 = \max\left[C'_1, \max_{P' \in S'} \left\{a' - (z'_o - G(P'))/W(P')\right\}\right]$$

$$x'_2 = \min\left[C'_2, \min_{P' \in S'} \left\{a' + (z'_o - G(P'))/W(P')\right\}\right].$$

Then the points $(x', y'_o)$ on the line segment determined by $(x'_1, y'_o)$ and $(x'_2, y'_o)$ are all optimal solutions to problem (5.6.1).

By Property 5.6.1, the optimal solution to the problem (5.2.1) is obtained by taking the inverse rotation, $T^{-1}$, of the solutions to (5.6.1). In Case 1, $(x_o, y_o) = T^{-1}(x'_o, y'_o)$ is the unique solution to problem (5.2.1). For Case 2, the points $(x_o, y_o)$ on the line segment determined by $(x_1, y_1) = T^{-1}(x'_o, y'_1)$ and $(x_2, y_2) = T^{-1}(x'_o, y'_2)$ are all optimal solutions to problem (5.2.1). Similarly for Case 3, the optimal solutions are on the line segment determined by the points $(x_1, y_1) = T^{-1}(x'_1, y'_o)$ and $(x_2, y_2) = T^{-1}(x'_2, y'_o)$.

In the above solution procedure for problem (5.2.1), there are several optimization problems to minimize or maximize some function over S or S × S, for example, determining $b_{st}$. Thus, obtaining the solution to problem (5.2.1) depends on being able to solve some optimization problems as subproblems. Assumptions have been made so that an optimal point to these functions exists;it is recognized that obtaining these optimal solutions may be difficult. The level of difficulty depends, of course, on the properties of the functions involved, that is, w(P), g(P), and on the properties of S. Chapter 2 shows that for the special case of S finite, these optimization problems are finite maximization or minimization problems and are readily solved.

# CHAPTER 6

# SUMMARY AND FUTURE RESEARCH

## 6.1 Summary

A study of several minimax location problems is presented in this dissertation. Each problem is to locate one or several new facilities in the plane with respect to existing facilities so as to minimize the maximum cost, where costs are specified functions of the rectilinear distance between new and existing facilities or among new facilities. For all but one of the problems considered, constraints are assumed that are upper bounds on the rectilinear distances between new and existing facilities.

For each problem considered, a formulation is given, possible applications are discussed, related literature is surveyed, properties of the problem are developed, and a solution procedure is provided.

For the problems considered in Chapters 2, 3 and 5, the development shows that a transformation, T, introduced in a study of location problems by Francis [15], applies to the respective problem and allows a decomposition into one-dimensional subproblems. The optimal solutions to the subproblems determine the optimal solution to the original problem.

Solution procedures are established for each subproblem that take advantage of the properties of rectilinear distance, the objective function, and other structures of the subproblem. In Chapters 2 and 5 a solution procedure is given for the one facility subproblem that expresses the unique optimal solution in terms of specified parameters of the subproblem; the parameters are specified by an expression that is determined for the optimal objective function value of each subproblem. In Chapter 3, the multifacility subproblem is written as an equivalent weighted network flow problem which is solved by determining flows of optimal "cost-to-weight" ratio in the network.

For the p-center problem, considered in Chapter 4, properties of the rectilinear distance enable the specification of a finite number of candidates for the optimal objective function values for all p-center solutions. An iterative solution procedure is provided that determines the optimal p-centers for all values of p. The transformation T is used in the computational development of the iterative procedure.

In Chapter 5, properties are presented of a one facility minimax location problem with respect to a compact set of existing facilities and using general distances. These properties extend similar results previously reported, [13], for a finite set of existing facilities.

## 6.2 Future Research

One possible direction for future work might be toward including more constraints on the problems considered. The solution procedures developed for the one facility and multifacility problems in Chapters 2 and 3 suggest that linear constraints on the location of the new facilities could be handled. Assuming linear constraints, the linear programming formulation of the transformed problem does not decompose into independent subproblems, but the constraints have block diagonal structure with coupling constraints corresponding to the transformed linear constraints. The subproblems in linear programming decomposition appear to share many of the properties of the subproblems obtained in Chapters 2 and 3. An investigation into the subproblems might lead to an efficient decomposition procedure for the linearly constrained facility location problem.

As mentioned in Chapter 4, the solution procedure for the p-center problem could be made more efficient if good bounds were known for the optimal objective function value of the p-center solution for all values of p. It would appear that the properties of the rectilinear distance would be useful in any investigation into these bounds.

Another area of research mentioned in Chapter 4 is an investigation into the p-center problem in one dimension. An efficient solution procedure, capitalizing on the structure of the one-dimensional problems, that improves on or bypasses the set covering problems, might lead to a more efficient algorithm for the p-center problem.

# REFERENCES

1 Beckenbach, E., and Bellman, R., An Introduction to Inequalities, Random House, New York, N. Y., 1961.

2 Cabot, A. V., Francis, R. L., and Stary, M. A., "A Network Flow Solution to a Rectilinear Distance Facility Location Problem," AIIE Transactions, Vol. 2, pp. 132-141, 1970.

3 Christofides, N., and Viola, P., "The Optimum Location of Multi-centers on a Graph," Operational Research Quarterly, Vol. 22, pp. 145-154, 1971.

4 Cooper, L., "Location Allocation Problems," Operations Research, Vol. 11, pp. 331-343, 1963.

5 Dantzig, G. B., Blattner, W., and Rao, M. R., "Finding a Cycle in a Graph with Minimum Cost to Time Ratio with Applications to a Ship Routing Problem," Theory of Graph International Symposium, Dunod, Paris, and Gordon and Breach, New York, N. Y., 1966, pp. 77-83.

6 Elzinga, J., and Hearn, D. W., "The Minimum Covering Sphere Problem," to appear in Management Science, 1972.

7 Elzinga, J., and Hearn, D. W., "Geometrical Solutions for Some Minimax Location Problems," to appear in Transportation Science, 1972.

8 Ford, L. B., Jr., and Fulkerson, D. R., Flows in Networks, Princeton University Press, Princeton, N. J., 1962.

9 Fox, B., "Finding Minimal Cost-Time Ratio Circuits," Operations Research, Vol. 17, pp. 546-550, 1969.

10 Francis, R. L., "A Note on the Optimum Location of New Machines in Existing Plant Layouts," Journal of Industrial Engineering, Vol. 14, pp. 57-59, 1963.

11 Francis, R. L., "On the Location of Multiple New Facilities with Respect to Existing Facilities," Journal of Industrial Engineering, Vol. 15, pp. 106-107, 1964.

12 Francis, R. L., "On Some Optimum Facility Design Problems," Ph.D. Dissertation, Northwestern University, Evanston, Ill., 1966.

13 Francis, R. L., "Some Aspects of a Minimax Location Problem," Operations Research, Vol. 15, pp. 1163-1169, 1967.

14 Francis, R. L., and Cabot, A. V., "Properties of a Multifacility Location Problem Involving Euclidean Distance," to appear in Naval Research Logistics Quarterly, 1972.

15 Francis, R. L., "A Geometrical Solution Procedure for a Rectilinear Distance Minimax Location Problem," to appear in AIIE Transactions, 1972.

16 Francis, R. L., "Minimax Rectilinear Distance Problems Involving Facilities Idealized as Points," unpublished working paper, University of Florida, Gainesville, Florida, 1971.

17 Garfinkel, R., and Nemhauser, G. L., "Optimal Set-Covering: A Survey," Perspectives on Optimization, edited by A. M. Geoffrion, Addison-Wesley Publishing Co., Reading, Mass., 1971, pp. 164-196.

18 Hakimi, S. L., "Optimum Distribution of Switching Centers in a Communication Network and Some Related Graph Theoretic Problems," Operations Research, Vol. 13, pp. 462-475, 1965.

19 Hearn, D. W., "Minimum Covering Spheres," Ph.D. Dissertation, Johns Hopkins University, Baltimore, Md., 1971.

20 Kuhn, H. W., "On a Pair of Dual Nonlinear Problems," Nonlinear Programming, Chapter 3, J. Abadie, editor, John Wiley & Sons, Inc., New York, N. Y., 1967.

21 Kuhn, H. W., "Location Problems and Mathematical Programming," Separatum-Colloqium on the Application of Mathematics to Economics, Budapest, 1963, Publishing House of the Hungarian Academy of Sciences, Budapest, 1965.

22 Kuhn, H. W., and Kuenne, R. E., "An Efficient Algorithm for the Numerical Solution of the Generalized Weber Problem in Spacial Economics," Journal of Regional Science, Vol. 4, pp. 21-33, 1962.

23 Love, R. F., "Locating Facilities in Three Dimensional Space by Convex Programming," Naval Research Logistics Quarterly, Vol. 16, pp. 503-516, 1969.

24 Love, R. F., Wesolowsky, G. O., and Kraemer, S. A., "A Multifacility Minimax Location Method for Euclidean Distances," to appear in International Journal of Production Research, 1972.

25 Ratliff, H. D., and Bellmore, M., "Set Covering and Involutory Bases," Management Science, Vol. 18, pp. 194-206, Nov., 1971.

26 Sylvester, J. J., Quarterly Journal of Mathematics, Vol. 1, p. 79, 1857.

27 Toregas, C., Swain, R., ReVelle, C., and Bergman, L., "The Location of Emergency Facilities," Operations Research, Vol. 19, pp. 1363-1373, 1972.

28 Wendell, R. E., "Some Aspects in the Theory of Location," Ph.D. Dissertation, Northwestern University, Evanston, Ill., 1971.

29 Wesolowsky, G. O., "Rectangular Distance Location Under the Minimax Optimality Criterion," Transportation Science, Vol. 6, pp. 103-113, 1972.

30 Wesolowsky, G. O., and Love, R. F., "The Optimal Location of New Facilities Using Rectangular Distances," Operations Research, Vol. 19, pp. 124-130, 1971.

# BIOGRAPHICAL SKETCH

The author, Perino Marcellin Dearing, Jr., was born in Atlanta, Georgia, on November 14, 1941. He received a Bachelor of Science degree with a major in mathematics from the University of North Carolina, Chapel Hill, North Carolina, in 1963, and a Master of Arts degree with a major in mathematics from the same university in 1965. The following four years he served as a commissioned officer in the United States Air Force. In September, 1969, he enrolled in the Graduate School of the University of Florida, Department of Industrial and Systems Engineering and received the Master of Engineering degree in 1971.

He is married to the former Kay Ledgerwood and has one son, William Ashley.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Richard L. Francis

Richard L. Francis, Chairman
Professor of Industrial and
Systems Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Donald W. Hearn

Donald W. Hearn, Co-Chairman
Assistant Professor of Industrial and
Systems Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Michael H. Moore

Michael H. Moore
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Hugh D. Ratliff

Hugh D. Ratliff
Assistant Professor of Industrial and
Systems Engineering

This dissertation was submitted to the Dean of the College of Engineering and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December, 1972

Robert E Uhrig

Dean, College of Engineering

Dean, Graduate School